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Weakly Union-free Twofold Triple Systems

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In memory of Paul Erdös

Abstract. In this paper, we settle a problem of Frankl and Füredi, which is a special case of a problem of Erdös, determining the maximum number of hyperedges in a 3-uniform hypergraph in which no two pairs of distinct hyperedges have the same union. The extremal case corresponds to the existence of weakly union-free twofold triple systems, which is settled here with six definite and four possible exceptions. An application to group testing is also given.

Keywords: union-free hypergraph, twofold triple system, group testing

1. Introduction

A *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:

(1) $\mathcal{G}$ is a partition of a set X (of *points*) into subsets called *groups*,
(2) $\mathcal{B}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point,
(3) every pair of points from distinct groups occurs in exactly λ blocks.

The parameter λ is the *index* of the GDD and $|X|$ is its *order*. The *group-type* (*type*) of the GDD is the multiset $[|G| : G \in \mathcal{G}]$. We usually use an "exponential" notation to describe group-type: a group-type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ denotes u_i groups of size g_i for $1 \leq i \leq s$. Groups of size 0 are permitted as a notational convenience. The type is *uniform* when all groups have the same size, in which case the type is of the form g^u.

If K is a set of positive integers, each of which is not less than 2, then we say that a GDD $(X, \mathcal{G}, \mathcal{B})$ is a K-GDD if $|B| \in K$ for every block B in $\mathcal{B}$. When $K = \{k\}$, we simply write k for K.

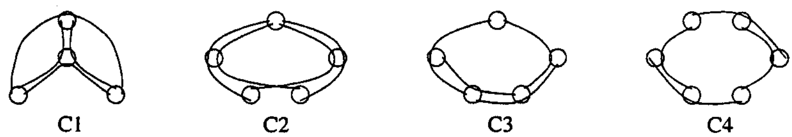
A 3-GDD $(X, \mathcal{G}, \mathcal{B})$ is a *weakly union-free* GDD (*wuf* GDD) if

(1) whenever $\{\{a,b,x\},\{a,b,y\}\} \subseteq \mathcal{B}$, the points x and y are in different groups, and
(2) whenever four distinct blocks B_1, B_2, B_3, B_4 are chosen from $\mathcal{B}$, it does not happen that $B_1 \cup B_2 = B_3 \cup B_4$.

The second condition can be made more explicit: there cannot exist four blocks of any of the following four forms:

$$\begin{aligned}
\text{C1}: &\quad \{\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\},\\
\text{C2}: &\quad \{\{x,a,b\},\{x,a,c\},\{x,b,d\},\{x,c,d\}\},\\
\text{C3}: &\quad \{\{x,a,b\},\{x,a,c\},\{x,b,d\},\{a,c,d\}\},\\
\text{C4}: &\quad \{\{x,a,b\},\{x,c,d\},\{y,a,b\},\{y,c,d\}\}.
\end{aligned}$$

These forms correspond, respectively, to the hypergraphs depicted below.



Our interest is in the construction of wuf 3-GDDs, and in particular, those of type 1^n and index two. A uniform GDD with group size 1 is a *balanced incomplete block design*; those with $k = 3$ and $\lambda = 2$ are called *twofold triple systems* of order n, or TTS(n). Frankl and Füredi [11] began the study of wuf TTS(n) in the study of an old problem of Erdös [10]. In 1938, Erdös [9] asked what the maximum number of edges a graph can have and have no 3-cycle, no 4-cycle, and no repeated edges. In 1977, he [10] asked the more general question: How many hyperedges can a k-uniform hypergraph have, so that whenever four hyperedges A, B, C, D satisfy $A \cup B = C \cup D$, we find $\{A, B\} = \{C, D\}$? Such a family is *union-free*. Frankl and Füredi [11] settled this question when $k = 3$, showing that a class of designs, the Steiner triple systems, realize the maximum.

They also addressed the related question of enforcing the union-free condition only for sets of four distinct blocks A, B, C, D. This gives the notion of weakly union-free, already defined. Frankl and Füredi [11] established an important bound and showed that it is realized infinitely often:

Theorem 1.1.[11] *A weakly union-free 3-uniform hypergraph on n vertices has at most $\lfloor n(n-1)/3 \rfloor$ hyperedges. Equality occurs when all, or all but one, pairs of vertices occur in two hyperedges each.*

They established that this bound is met whenever $n \equiv 1 \pmod 6$, and either n is a prime power at least 13 or n is sufficiently large. In this paper, we establish that equality is met for all $n \equiv 0, 1 \pmod 3$, with a small number of definite and a small (mod 3). While we have also found small designs sufficient to obtain a closure in this class, we concentrate on the twofold triple system case here.

The difficulty of this problem appears initially to be that, while catalogues of twofold triple systems for small orders are available (see [5], for example), no TTS(n) is weakly union-free when $n \in \{3,4,6,7,9,10\}$. Moreover, when a wuf 3-GDD of type T can be decomposed into two 3-GDDs of index one and type T, condition (1) together with the exclusion of C4 ensure that these two index one 3-GDDs are "orthogonal" (see [6]). The existence of orthogonal uniform 3-GDDs with group size 1, the so-called orthogonal Steiner triple systems, remained open for thirty years until its recent solution [7]. The exclusion of further configurations adds to the difficulty of the problem for wuf TTS.

2. Direct Constructions

In this section, we develop a direct construction technique that is used to construct both wuf TTS(n) and, more generally, wuf 3-GDDs of index two. The general framework follows. We aim to construct a 3-GDD of index two on point set $\mathbb{Z}_{gu} \cup \{\infty_1,\ldots,\infty_k\}$, whose type is $g^u k^1$. Naturally, we chose $\mathbb{Z}_{gu}$ for a portion of the point set to suggest the cyclic action of the cyclic group on these points. Indeed our goal is to construct 3-GDDs that have $\mathbb{Z}_{gu}$ as an automorphism group.

Let $X = \mathbb{Z}_{gu} \cup \{\infty_1,\ldots,\infty_k\}$ and σ a permutation mapping $i \mapsto i+1 \bmod gu$ for $i \in \mathbb{Z}_{gu}$, and fixing $\{\infty_1,\ldots,\infty_k\}$. Let $\mathcal{B}$ be the blocks of a 3-GDD of type $g^u k^1$ on X that admits σ as an automorphism. The action of σ partitions $\mathcal{B}$ into orbits of size gu or, when $gu \equiv 0 \pmod 3$, possibly $gu/3$. A set of representatives of these orbits forms a *set of starter blocks* for the 3-GDD. Starter blocks of the form $\{0,a,b\} \subset \mathbb{Z}_{gu}$ may generate orbits of length gu under σ, in which case the starter block is said to *cover* the differences $\pm a$, $\pm b$, $\pm(b-a)$ with arithmetic in $\mathbb{Z}_{gu}$ (if repetitions occur, such differences are covered the number of times that they occur). When $gu \equiv 0 \pmod 3$, a starter block of the form $\{0, gu/3, 2(gu/3)\}$ generates only $gu/3$ distinct blocks (a *short orbit*), and is therefore said to *cover* the differences $\pm gu/3$ once each. Finally, a starter block may have the form $\{\infty_i, 0, d_i\}$. Again, gu blocks appear in the orbit generated, but here only the differences $\pm d_i$ are covered once each.

A set $\mathcal{D}$ is a set of starter blocks for a 3-GDD of index two and type $g^u k^1$ (under the action of σ) if

(1) for $1 \le i \le k$, there is exactly one starter block containing ∞_i, and
(2) each $d \in \mathbb{Z}_{gu}$ is covered twice as a difference, unless $d \equiv 0 \pmod u$, in which case the difference is not covered.

The reader can quickly verify that these conditions on starter blocks are equivalent to the existence of a 3-GDD of index two and type $g^u k^1$ admitting σ.

In order to be a wuf 3-GDD, further conditions are imposed. Suppose $\mathcal{D}$ is the set of starter blocks for a 3-GDD of index two and type $g^u k^1$. Partition $\mathcal{D}$ into the blocks $\mathcal{A}$ which contain one of the infinite points, and the blocks $\mathcal{B}$ which do not. Evidently, $\mathcal{A}$ contains exactly k blocks, one for each of the infinite points. In addition, in order to meet the first wuf condition, we have:

(1) If $\{\infty_i, 0, a\}, \{\infty_j, 0, b\} \in \mathcal{A}$, then $a \not\equiv \pm b \pmod{gu}$.

Call a difference *external* if it is covered once in $\mathcal{A}$ and once in $\mathcal{B}$, and *internal* if it is covered twice in $\mathcal{B}$. For each external difference d, define $\alpha(d) = \min(\pm 2d)$. For each internal difference d, when blocks $\{0,d,x\}$ and $\{0,d,y\}$ appear in the orbits of blocks of $\mathcal{B}$, define $\alpha(d) = \min(\pm(x-y))$.

First, we examine constraints resulting from prohibiting the appearance of one of the infinite points in one of the configurations C1, C2, C3, or C4. In order to ensure that no infinite points occur in a C1 configuration, we require that

(2) If $gu \equiv 0 \pmod 3$ and $gu/3$ is an external difference, then $\mathcal{B}$ does not contain $\{0, gu/3, 2(gu/3)\}$.

In order to ensure that no infinite points occur in a C2 configuration, we require that

(3) If d is an external difference, then $4d \not\equiv 0 \pmod{gu}$.

In order to ensure that no infinite points occur in a C3 configuration, we require that

(4) If d is an external difference and $\{0,d,x\}$ is a block in an orbit of a starter block of $\mathcal{B}$, then $2x \not\equiv d \pmod{gu}$ and none of $\{0,d,3d\}$, $\{0,2d,3d\}$, $\{0,2d,d+x\}$, $\{0,2d,x\}$, or $\{0,d,gu/2\}$, when $gu \equiv 0 \pmod 2$, appear in the orbits of the starter blocks in $\mathcal{B}$.

In order to ensure that no infinite points occur in a C4 configuration, we require that

(5) If d and d' are external differences, or if d is external and d' is internal, then $\alpha(d) = \alpha(d')$ only if $d = d'$.

Once conditions (1) – (5) are met, any violation of the wuf conditions occurs entirely among the blocks on $\mathbb{Z}_{gu}$.

In order to check that none of the conditions are violated on the blocks involving no infinite points, we first observe that the first wuf condition is equivalent to:

(6) If d is an internal difference, then $\alpha(d) \not\equiv 0 \pmod u$.

To check that the four configurations are missed, we first form the restricted neighborhood of the point 0, defined by $N_0 = \{\{x,y\} : \{0,x,y\}$ is in the orbit of a block of $\mathcal{B}\}$. This is a graph on the vertex set consisting of $\mathbb{Z}_{gu}$, except integers congruent to 0 (mod u). In this graph, every vertex has degree one or two, and so N_0 consists of disjoint paths and cycles.

To avoid C1, we require:

(7) If (a,b,c) forms a 3-cycle in N_0, then $\{a,b,c\}$ does not appear in the orbit of any block in $\mathcal{B}$.

To avoid C2, we require:

(8) N_0 does not contain a 4-cycle.

To avoid C3, we require:

(9) For every 3-edge path (a,b,c,d) in N_0, no block of the form $\{a,c,d\}$ or $\{a,b,d\}$ appears in the orbit of a block of $\mathcal{B}$.

To avoid a C4 configuration, we require that

(10) If d and d' are internal differences, then $\alpha(d) = \alpha(d')$ only if $d = d'$.

This is apparently an extensive list of conditions, but each condition is easily checked. In fact, Frankl and Füredi [11] establish the existence of some GDDs of index 2 and type 1^q when $q \equiv 1 \pmod 6$. They show that if ω is a primitive element of the finite field $\mathbb{F}_q$ and $q \equiv 1 \pmod 6$, $q \neq 7$, then $\{\{0, \omega^i, \omega^{2t+i} + \omega^i\} : 0 \leq i < 2t\}$ is a set of starter blocks for a wuf TTS(q). Hence, we have:

Lemma 2.1.[11] *A 3-GDD of index two and type 1^q exists whenever $q \equiv 1 \pmod 6$ is a prime power, except when $q = 7$.*

It is essential that ingredients for other congruence classes modulo 6 be found as well. We employed a combination of backtracking and hillclimbing techniques to produce a large number of wuf GDDs.

Numerous 3-GDDs of type $1^u x^1$ over $\mathbb{Z}_u$ are given in order to establish the statement:

Lemma 2.2. *A wuf 3-GDD of type 1^n exists for $n = 21$, 24, 27, 28, 30, 33, 34, 36, 39, 40, 42, 45, and 46.*

Proof. For each pair $\{a,b\}$ presented in the table to follow, $\{0,a,b\}$ is a starter block. In addition, if $u \equiv 0 \pmod 3$ and $x \equiv 1 \pmod 3$, then $\{0, u/3, 2u/3\}$ is a starter block. Finally, each difference covered only once in the starter blocks so produced is also in a starter block with an infinite point.

GDD	Internal Starter Blocks
$1^{20}1^1$	{1,7} {1,9} {2,4} {3,8} {3,13} {4,9}
$1^{23}1^1$	{1,6} {2,13} {2,16} {3,12} {3,18} {4,8} {6,16}
$1^{26}1^1$	{1,6} {2,12} {2,23} {3,19} {4,13} {4,18} {6,17} {7,18}
$1^{28}0^1$	{1,2} {2,13} {3,7} {3,12} {4,12} {5,11} {5,19} {6,13} {8,18}
$1^{29}1^1$	{1,7} {2,15} {2,18} {3,8} {3,12} {4,22} {4,23} {5,19} {8,17}
$1^{32}1^1$	{1,3} {1,8} {3,10} {4,19} {4,20} {5,18} {5,26} {6,23} {8,18} {9,21}
$1^{33}1^1$	{1,4} {2,8} {2,21} {3,16} {4,26} {5,15} {5,24} {6,24} {7,23} {8,21}
$1^{35}1^1$	{1,4} {2,6} {2,19} {3,20} {5,12} {5,29} {7,18} {8,16} {9,22} {9,23} {10,20}
$1^{38}1^1$	{1,4} {2,7} {2,17} {3,15} {4,18} {5,13} {6,12} {7,27} {8,22} {9,22} {9,28} {10,27}
$1^{40}0^1$	{1,2} {2,5} {3,7} {4,27} {5,15} {6,24} {7,29} {8,21} {8,28} {9,21} {9,26} {11,26}
$1^{41}1^1$	{1,4} {2,6} {2,14} {3,30} {5,24} {5,31} {6,21} {7,18} {7,32} {8,20} {8,25} {9,28} {10,28}
$1^{44}1^1$	{1,2} {2,5} {3,7} {5,13} {6,20} {6,33} {7,23} {8,24} {9,26} {9,19} {10,34} {11,31} {12,30} {13,28}
$1^{45}1^1$	{1,4} {2,6} {2,7} {3,29} {5,35} {6,34} {7,25} {8,22} {8,29} {9,20} {9,31} {10,27} {12,24} {13,26}

∎

Lemma 2.3. *A wuf 3-GDD of type 1^n exists for n =* 48, 51, 52, 54, 55, 57, 58, 60, 63, 64, 66, 69, 70, 72, 75, 76, 78, 81, 82, 84, 85, 87, 88, 90, 91, 93, 94, 96, 99, 100, 102, 105, 108, 111, 112, 114, 115, 117, 118, 120, 123, 124, 126, 129, 130, 132, 133, 135, 136, 138, 141, 142, 144, 145, 148, 150, 154, 156, 159, 160, 161, 165, 166, 171, 177, 178, 184, 195, 201, 207, 213, 219, *and* 243.

Proof. Constructions are available from the authors. ∎

The remaining small values do not appear to be able to be handled by this general approach. However, we have succeeded in one more case.

Lemma 2.4. *A wuf 3-GDD of type 1^{16} exists.*

Proof. Let $X = \mathbb{Z}_8 \times \{0,1\}$. For succinctness, we write $(x,i) \in X$ as x_i. Let $\sigma : X \to X$ be the permutation such that $\sigma : x_i \mapsto (x+1 \pmod 8)_i$. Developing the following set of starter blocks by σ gives a wuf 3-GDD of type 1^{16} on X:

$$\begin{array}{lllll} \{0_0,1_0,3_1\} & \{0_0,4_0,0_1\} & \{0_0,2_0,5_0\} & \{0_0,2_0,1_1\} & \{3_0,0_1,1_1\} \\ \{0_0,1_1,3_1\} & \{0_0,1_0,5_1\} & \{0_1,2_1,5_1\} & \{0_0,2_1,6_1\} & \{0_0,0_1,7_1\}. \end{array}$$ ∎

3. Recursive Constructions

We employ two well-known constructions.

Theorem 3.1.[15] *Let $(X, \mathcal{G}, \mathcal{B})$ be a GDD (the* master *GDD) with groups $G_1, G_2 \ldots G_t$. Suppose there exists a function $w : X \to \mathbb{Z}^+ \cup \{0\}$ (a* weight function*) which has the property that, for each block $B = \{x_1, x_2, \ldots, x_k\} \in \mathcal{B}$, there exists a K-GDD of type $[w(x_1), w(x_2) \ldots, w(x_k)]$ (such a GDD is an* "ingredient" *GDD). Then there exists a K-GDD of type*

$$\left[\sum_{x\in G_1} w(x), \sum_{x\in G_2} w(x), \ldots, \sum_{x\in G_t} w(x)\right].$$

We leave as an easy exercise that when all of the ingredient GDDs are wuf, so is the GDD constructed. In general, our desire is to produce GDDs with group size 1, so we need to fill in the holes in some way.

Theorem 3.2.[12] *If there exists a wuf GDD of type $g_1g_2\cdots g_n$, and for $2 \le i \le n$, a wuf GDD of type $1^{g_i}h^i$ exists, then there exists a wuf GDD of type $1^{\sum_{i=2}^{n} g_i}(g_1+h)^1$.*

In Theorem 3.2, both $g_1 = 0$ and $h = 0$ correspond to useful special cases. Filling in holes preserves the wuf property primarily as a consequence of the first requirement, since none of the forbidden configurations can have both a block from the wuf GDD of type $g_1g_2\cdots g_n$ and one from a wuf GDD of type $1^{g_i}h^i$. Normally, we do not comment on applications of Theorem 3.2, leaving this to the diligent reader. Typically, Theorem 3.1 is applied using suitable ingredients, and Theorem 3.2 is then applied to extract useful consequences for group size 1.

The master designs used in Theorem 3.1 all arise from the same source. A *transversal design $TD(k,n)$* is a k-GDD of type n^k. There is an extensive literature on the existence of transversal designs [1], but for our purpose here, one result suffices:

Lemma 3.3.[1] *If q is a prime power and $1 \leq k \leq q+1$, then a $TD(k,q)$ exists.*

Now we give some applications of Theorem 3.1.

Lemma 3.4. *If a $TD(6,n)$ exists, then a wuf 3-GDD of type $(3n)^5(6n)^1$ exists. Moreover, there exist wuf TTS of orders* 106, 147, 168, 189, *and* 231.

Proof. A wuf 3-GDD of type $3^5 6^1$ exists with presentation $\{\{1,12\},\{2,9\}\}$. Use the $TD(6,n)$ as a master design and the 3-GDD of type $3^5 6^1$ as an ingredient design in Theorem 3.1. Apply with $n = 5,7,8,9,11$ and fill in holes using wuf 3-GDDs of types $1^{15}1^1$ and $1^{30}1^1$ when $n = 5$, and of types 1^{3n} and 1^{6n} for the remaining values of n. ∎

Lemma 3.5. *If a $TD(7,n)$ exists, then a wuf 3-GDD of type $(2n)^7$ exists. Hence, wuf TTS of orders* 112, 183, *and* 225 *exist.*

Proof. One wuf 3-GDD of type 2^7 has presentation $\{\{1,4\},\{1,6\},\{2,6\},\{2,11\}\}$; Theorem 3.1 gives the wuf 3-GDD of type $(2n)^7$. Applying with $n = 8,13,16$, and filling holes with wuf 3-GDDs of types 1^{16}, $1^{26}1^1$, and $1^{32}1^1$ gives the required consequences. ∎

Lemma 3.6. *If a $TD(8,n)$ exists and $0 \leq x \leq n$, then a wuf 3-GDD of type $(3n)^7(3n+6x)^1$ exists. Hence, there exist wuf TTS of orders* 174, 180, 186, 192, 198, 204, 210, 216, 222, 228, *and* 187.

Proof. A wuf 3-GDD of type $3^8 = 3^7 3^1$ exists over $\mathbb{Z}_{24}$ with presentation

$$\{\{1,3\},\{1,20\},\{2,12\},\{3,10\},\{4,11\},\{5,18\},\{6,15\}\}.$$

A wuf 3-GDD of type $3^7 9^1$ exists with presentation $\{\{1,13\},\{2,5\},\{4,10\}\}$. Apply Theorem 3.1 using weight 3 in seven groups and weight 3 or 9 in the eighth, to produce a wuf 3-GDD of type $(3n)^7(3n+6x)^1$. Apply with $n = 7,8$ and fill in holes to obtain the stated consequences. For the final value, apply with $n = 7$ and employ a wuf 3-GDD of type $1^{21}7^1$ to fill holes. It has presentation $\{\{1,4\},\{1,6\},\{2,9\},\{2,13\}\}$. ∎

Lemma 3.7. *If a $TD(14,q)$ exists and $0 \leq x \leq 6q$ satisfies $x \equiv 0 \pmod 3$, then a wuf 3-GDD of type $q^{13}x^1$ exists. If, in addition, a wuf 3-GDD of type $1^q h^1$ exists, so does a wuf GDD of type $1^{13q}(x+h)^1$.*

Proof. Use as ingredient wuf 3-GDDs the ones of type $1^{13}0^1$ from Lemma 2.1, of type $1^{13}3^1$ presented as $\{\{1,4\},\{3,4\},\{2,8\}\}$, and of type $1^{13}6^1$ presented as $\{\{1,4\},\{2,8\}\}$. Give all points in thirteen groups of the $TD(14,q)$ weight one, and points in the final group weights 0, 3, or 6 so that the total weight in the final group is x. Theorem 3.1 then gives a wuf 3-GDD of type $q^{13}x^1$. Filling in holes with a 3-GDD of type $1^q h^1$ (when one exists) gives a wuf GDD of type $1^{13q}(x+h)^1$. ∎

Corollary 3.8. *A wuf TTS(n) exists whenever $n = 169$ or $190 \leq n \leq 304$ and $n \equiv 1 \pmod 3$.*

Proof. If $n \leq 247$, write $n = 13 \cdot 13 + x$, then $0 \leq x \leq 6 \cdot 13$ and a wuf 3-GDD of type 1^x has been previously given. Apply Theorem 3.7 with a $TD(14,13)$. If instead $n > 247$, write $n = 13 \cdot 16 + x$ and proceed similarly with a $TD(14,16)$. ∎

Corollary 3.9. *A wuf TTS(n) exists whenever $n \equiv 0 \pmod 3$ and $234 \le n \le 327$, except possibly when $n = 243$.*

Proof. Write $n = 13 \cdot 17 + 4 + x$. A wuf 3-GDD of type 1^{x+4} has been presented, and a wuf 3-GDD of type $1^{17}4^1$ has presentation $\{\{1,3\},\{1,5\},\{2,12\},\{3,11\}\}$. Apply Lemma 3.7 with $h = 4$. ∎

Lemma 3.10. *If a $TD(k,q)$ exists for $k \ge 14$, a wuf 3-GDD of type 1^q exists, and $0 \le x \le 6(k-13)$ satisfies $x \equiv 0 \pmod 3$, then a wuf 3-GDD of type $1^{13(q-1)}(x+13)^1$ exists. Hence, wuf TTS of orders 172 and 175 exist.*

Proof. Use the same ingredients as Lemma 3.7. Give all points in thirteen groups of the $TD(14,q)$ weight one, and choose a single block B of length k. Assign all points in the remaining $k-13$ levels weight 0 if they are *not* on B, and weight 0, 3, or 6 if they are on B so that the total weight of such points is x. When $q = 13$ and $x = 3,6$, wuf 3-GDDs of types $1^{156}16^1$ and $1^{156}19^1$ result. ∎

Theorem 3.11. *Let $n \equiv 0,1 \pmod 3$. If wuf TTS exist for all orders n satisfying $24 \le n \le 304$, then wuf TTS exist for all orders $n \ge 24$.*

Proof. Form two infinite sequences of integers $(r_i : i \ge 0)$ and $(s_i : i \ge 0)$ for which

(1) $r_0 = 19$ and $s_0 = 23$;
(2) $r_{i+1} > r_i$ and $s_{i+1} > s_i$;
(3) $13r_{i+1} + 21 \le 19r_i$ and $13s_{i+1} + 25 \ge 19s_i + 1$;
(4) $r_i \equiv 1 \pmod 3$ and $s_i \equiv 2 \pmod 3$, and
(5) $TD(14, r_i)$ and $TD(14, s_i)$ exist.

A $TD(14,n)$ exists whenever n is relatively prime to 2, 3, 5, 7, and 11 (by MacNeish's theorem; see [1]). Among the integers congruent to 1 modulo 3, considering the sequence of those relatively prime to 2, 3, 5, 7, and 11, we find a largest difference between consecutive values of 24. Choose the r_i's to be the sequence of numbers congruent to 1 modulo 3 and relatively prime to 2, 3, 5, 7, and 11, beginning with 19, in addition to the number 25. It is now an easy verification that we have the specified properties. In the same way, the s_i's are the sequence of numbers congruent to 2 modulo 3 and relatively prime to 2, 3, 5, 7, and 11, beginning with 23, in addition to the number 32.

To prove the theorem, we proceed inductively. In general, we suppose wuf TTS have been produced for all orders less than n, where $n \equiv 0,1 \pmod 3$, and we establish that a wuf TTS(n) exists. By assumption, wuf TTS(n) exist whenever $24 \le n \le 304$. Now, if $n \equiv 1 \pmod 3$, find the largest i for which $13r_i + 24 \le n \le 19r_i$, such a choice exists by the definition of the sequence. Then a $TD(14, r_i)$ exists. Wuf 3-GDDs of type 1^{r_i} and 1^{n-13r_i} exist by the inductive hypothesis. Apply Theorem 3.7 to obtain the wuf TTS(n). In the same way, if $n \equiv 0 \pmod 3$, find the largest s_i for which $13s_i + 25 \le n \le 19s_i + 1$; such a choice exists by the definition of the sequence. Then a $TD(14, s_i)$ exists. Wuf 3-GDDs of type $1^{s_i}1^1$ and 1^{n-13s_i} exist by the inductive hypothesis. Apply Theorem 3.7 to obtain the wuf TTS(n). ∎

Now we can prove the main theorem.

Theorem 3.12. *A wuf TTS(n) exists whenever* $n \equiv 0,1 \pmod 3$ *except when* $n \in \{3,4,6,7,9,10\}$ *and possibly when* $n \in \{12,15,18,22\}$.

Proof. The definite exceptions can all be verified by an exhaustive search. Now, if n is a prime or prime power, apply Lemma 2.1. Otherwise, apply Lemmas 2.2, 2.3, and 2.4 to treat most small orders, and Lemmas 3.4, 3.5, 3.6, 3.10 and Corollaries 3.8 and 3.9 to treat $n = 21$ and all remaining values satisfying $24 \leq n \leq 304$. Then apply Theorem 3.11 to complete the proof. ∎

4. An Application to Group Testing

Let Ω be a population of items, where each item is in exactly one of the states 0, 1. Furthermore, at most r items are in state 1. The problem is to determine the state of each item (or equivalently, to determine the set of all items in state 1) through some tests. A test can be performed on any subset $P \subseteq \Omega$, called a *pool*. The *feedback* to a test on pool P, denoted $f(P)$, is defined by $f(P) = \max_{\omega \in P}\{\text{state of } \omega\}$. This problem, known as the *group testing problem*, has numerous real-world applications ranging from multiple access communications [2] to DNA clone isolation [4], and its study constitutes an important part of combinatorial search theory [8]. In some applications, it is desirable to have each item involved in exactly k pools. We call the resulting problem *k-restricted*. For simplicity, we denote the k-restricted group testing problem, with at most r items in state 1, by $\mathrm{GTP}_k(r)$.

An algorithm for the group testing problem is said to be an *α-approximation algorithm* if it returns a set S of at most αr items, so that S contains all items of Ω that are in state 1.

There are two well-known classes of algorithms for solving group testing problems: *sequential* and *nonadaptive* algorithms. In a sequential algorithm, the decision of which pool to test next can depend on the feedback to previous tests. On the other hand, a nonadaptive algorithm must specify all the pools to be tested at the very beginning, without receiving any feedback. The *complexity* of a group testing algorithm is defined to be the number of tests conducted (hence, also the number of pools). The best sequential algorithm has a complexity no higher than any nonadaptive algorithm. However, the advent of massively parallel computers prompted Hwang and Sós [13] to make a case for the study of nonadaptive algorithms. Further motivation is given by Knill and Muthukrishnan [14] who observed that certain features in the screening of clone libraries with hybridization probes strongly encourage nonadaptive algorithms.

Our focus in this section is on nonadaptive 3/2-approximation algorithms for $\mathrm{GTP}_3(2)$. Any nonadaptive algorithm $\mathcal{A}$ for $\mathrm{GTP}_3(2)$ corresponds to a 3-uniform hypergraph $\mathcal{H}(\mathcal{A}) = (X, \mathcal{B})$ as follows:

(1) $X = \{x_P : P \text{ is a pool of } \mathcal{A}\}$.

(2) $\mathcal{B} = \{B_\omega : \omega \in \Omega\}$.

(3) $x_P \in B_\omega$ if and only if $\omega \in P$.

We call $\mathcal{H}(\mathcal{A})$ the *hypergraph of* $\mathcal{A}$. We make the following useful observation concerning $\mathcal{H}(\mathcal{A})$. Let O be the set of all state 1 items in Ω. Then $x_P \in \bigcup_{\omega\in O} B_\omega$ if and only if P is a pool of $\mathcal{A}$ such that $f(P) = 1$. Hence, if we know that one of O or O' contains the set of all state 1 items in Ω, then a necessary and sufficient condition which allows us to distinguish them is

$$\bigcup_{\omega\in O} B_\omega \neq \bigcup_{\omega\in O'} B_\omega.$$

Lemma 4.1. *If $\mathcal{A}$ is a nonadaptive 3/2-approximation algorithm for $GTP_3(2)$, then $\mathcal{H}(\mathcal{A}) = (X,\mathcal{B})$ is weakly union-free.*

Proof. Assume to the contrary that there are four distinct hyperedges $B_{\omega_i} \in \mathcal{B}$, $1 \le i \le 4$, such that $B_{\omega_1} \cup B_{\omega_2} = B_{\omega_3} \cup B_{\omega_4}$. If one of $\{\omega_1,\omega_2\}$ or $\{\omega_3,\omega_4\}$ is the pair of state 1 items, then $\mathcal{A}$ cannot distinguish them. The best $\mathcal{A}$ can do is then to conclude that $\{\omega_1,\omega_2,\omega_3,\omega_4\}$ contains all state 1 items of Ω. But this violates the condition that $\mathcal{A}$ is a 3/2-approximation algorithm. ∎

Lemma 4.2. *The complexity of any nonadaptive 3/2-approximation algorithm for $GTP_3(2)$ with a population of n items is at least $\lceil\sqrt{3n} + (1/2)\rceil$.*

Lemma 4.3. *Any wuf $TTS(n)$ is the hypergraph of a nonadaptive 3/2-approximation algorithm for $GTP_3(2)$.*

Proof. Let $\mathcal{A}$ be the nonadaptive algorithm specified by a wuf TTS(n), $\mathcal{H}(\mathcal{A}) = (X,\mathcal{B})$. Let $\omega_1,\omega_2,\omega_3 \in \Omega$ be any three distinct items. Then $B_{\omega_1} \neq B_{\omega_2}$ since $H(\mathcal{A})$ contains no repeated hyperedges, and $B_{\omega_1} \neq B_{\omega_2} \cup B_{\omega_3}$ since the union of two distinct hyperedges contains at least four vertices. Hence, if Ω contains only one item in state 1, then $\mathcal{A}$ can identify that item precisely. We are thus left with the task of considering the case with two items in state 1.

It suffices to show that for any three distinct hyperedges $B_{\omega_1},B_{\omega_2},B_{\omega_3} \in \mathcal{B}$ such that $B_{\omega_1} \cup B_{\omega_2} = B_{\omega_1} \cup B_{\omega_3} = F$, we have $\{B,B'\} \subseteq \{B_{\omega_1},B_{\omega_2},B_{\omega_3}\}$ whenever $B \cup B' = F$. So let $B \cup B' = F$. Suppose that at least one of B or B' is not B_{ω_1}, B_{ω_2}, or B_{ω_3}, otherwise we are done. Therefore, we must have $\{B,B'\} = \{B_{\omega_1},B_{\omega_4}\}$, for some $\omega_4 \notin \{\omega_1,\omega_2,\omega_3\}$ since $(X,\mathcal{B})$ is weakly union-free. We know that $|B_{\omega_1} \cap B_{\omega_2}| \neq 0$ or 3 because $\mathcal{B}$ contains no repeated hyperedges. If $|B_{\omega_1} \cap B_{\omega_2}| = 2$, then $|F| = 4$, implying that $\{B_{\omega_1},B_{\omega_2},B_{\omega_3},B_{\omega_4}\}$ is the complete 3-uniform hypergraph on four vertices, which is not weakly union-free. It follows that $|B_{\omega_1} \cap B_{\omega_2}| = 1$. But then $B_{\omega_2} \setminus B_{\omega_1}$ is a 2-subset that must also be contained in the blocks B_{ω_3} and B_{ω_4}. This contradicts the assumption that $(X,\mathcal{B})$ is a twofold triple system. ∎

Lemma 4.4. *For any $n \equiv 0,1 \pmod 3$, and $n > 22$, there exists a nonadaptive 3/2-approximation algorithm of (optimal) complexity n for $GTP_3(2)$ with a population of $n(n-1)/3$ items.*

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