Thermal-Aware Communication

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***Abstract*—Temperature control is of utmost importance in transmission systems. In this paper, a binary channel model is considered in which the transmission of a one causes a temperature increase while communicating a zero causes a temperature drop. By putting constraints on the input sequences, it is guaranteed that the channel temperature will not exceed a certain pre-determined maximum. In the asymptotic regime, the capacity of such a channel is studied. For the non-asymptotic regime, fixed-length codes are presented, with the property that codewords can be freely cascaded without violating the temperature constraint. Optimization of the code size is investigated and codewords are enumerated using generating functions.**

***Index Terms*—Constrained codes, thermal-aware channel, communication.**

I. Introduction

Power and heat dissipation have emerged as first-order design constraints for chips, whether targeted for battery-powered devices or high-end systems. High temperatures have dramatic negative effects on bus performance. Power-aware design alone is insufficient to address the thermal challenges since it does not directly target the spatial and temporal behavior of the operating environment. For this reason, thermally aware approaches have emerged as one of the most important domains of research in chip design today. Numerous techniques have been proposed to reduce the overall power consumption of on-chip buses (see [2], [3] which uses coding techniques and the references therein using non-coding techniques). All the non-coding techniques do not directly address peak temperature minimization. The coding techniques such as in [2] and [3] assume that the ℓ wires that have the highest temperature are known to the transmitter, but it is not elaborated how this is known to the transmitter. Moreover, it is not known if ℓ is indeed the number of wires that are in danger of overheating. The goal of the current paper is to take one step to bridge this gap in the coding scheme. We will analyze the properties of one wire and apply it later to each wire from a multi-wire device, a topic that is beyond the current work.

An example of a constrained binary channel which consists of only one wire that accepts only one bit at a time slot is a laser diode. Laser diodes have been widely used in optical communications and data storage, both optical and heat-assisted magnetic recording [4]. For example, in a binary recording channel, a high-power laser diode is used to record (burn) data into an optical disc [5]. In numerous channels, data are recorded or transmitted by switching such a laser on and off. As temperature increases, the efficiency of the laser decreases, and as a result, more drive current is required to turn on the laser diode [6]. In order to reduce the dissipation and extend the life of the electronic component, prior art code design has been guided by minimizing the average number of laser pulses [7].

In order to make communication systems function properly, it is of great importance that their electronic components do not overheat. In order to avoid overheating electronic devices, it is possible to use some coding scheme to control the temperature of electronic components. In literature, there are various techniques, see [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], and [15], that have been proposed over the years in order to deal with temperature control. In this work, we aim to propose a new coding scheme to control peak-temperature of electronic devices with following assumption. The assumption is made that sending a *one* comes with a temperature rise, while a zero leads to a cooling down. This can also be interpreted in another way, changing the information from *zero* to *one* or from *one* to *zero* increases the temperature. If the information does not change, the temperature decreases. Another interpretation can be that having a pulse (an 'on' state) implies that the information is a *one*, while no pulse (an 'off' state) in the same slot of time means that a *zero* was sent. Each binary system can have its

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own features to simulate this scenario. Our coding scheme is based on constraining the input sequences to guarantee that the system temperature does not exceed a certain maximum value during the transmission of binary digits. We investigate the channel, called thermal-aware channel, that only accepts these input sequences.

Analyzing the capacity of the channel and designing coding schemes with this channel is based on the basic parameters and properties of the channel. There are a few such parameters. The first parameter is the lowest possible temperature of the channel which w.l.o.g. (without loss of generality) can be taken as 0 and hence it can be usually ignored. This temperature is also the initial temperature of the channel. The second parameter is the highest temperature of the channel T_{max} which for simplicity will be denoted by T. Above this temperature, the electric instrument (laser, wire, etc.) will be burned. From now on it will always be called a wire. The third parameter is t_1, the ***heating gradient***. Whenever either the information on the channel is changed or a pulse is sent, the wire will be heated by t_1 and it should not exceed T_{max} as otherwise it will be burned. The fourth parameter is the ***cooling gradient*** t_0. Whenever the information on the channel remains unchanged or a pulse is not sent in a certain slot of time, the temperature is decreased by t_0, but not below the initial temperature 0.

In this work, we study thermal-aware channels: their properties, their capacities, and constructions of codes. More specifically, the contributions of this paper are as follows.

- We show that the proposed thermal-aware channel is a constrained channel which is closely related to other constrained channels.
- We present several techniques to compute the capacity of the channel. In general cases, we provide several bounds on the capacity based on their relations with other constrained channels. In some cases, we obtain an explicit expression of the capacity of the channel. We also present numerical results in various cases, especially when all parameters are small.
- We design fixed-length codes with the property that codewords can be freely cascaded without exceeding the maximum temperature.
- We optimize the rates of the codes and use generating functions in order to enumerate the codewords.

The rest of this paper is organized as follows. In Section II we describe the thermal-aware channel model under consideration. Then, in Section III, properties of thermal-aware sequences are provided, and their relations with running digital sum and d-constrained sequences are given. Next, Section IV investigates the capacity of the thermal-aware channel. Coding techniques are presented and analyzed in Section V. Finally, conclusions are drawn in Section VI.

II. Thermal-Aware Channel Model

We start with a detailed description of the thermal-aware channel model. It is a binary noiseless channel with a constraint on its maximum temperature. In every time slot either a zero or a one is transmitted. It is assumed that the transmission of a one increases the channel temperature, due to physical aspects caused by, e.g., sending a pulse. On the other hand, it is assumed that communicating a zero causes a temperature decrease, for example since no physical transmission at all takes place in such a time slot. We assume that the temperature does not drop below a certain base temperature.

The temperature increase due to communicating a one is called the *heating gradient* and is denoted by t_1. The temperature decrease due to communicating a zero is called the *cooling gradient* and is denoted by t_0. The ratio of t_1 and t_0 is denoted by k. Assuming that t_1 and t_0 are positive rational numbers, we can uniquely write this ratio as

$$\frac{t_1}{t_0} = k = \frac{p}{q}, \tag{1}$$

where p and q are positive co-prime integers. The channel base temperature is $T_{\min}$ and the maximum allowed temperature is $T_{\max}$. Let the width of the temperature range $[T_{\min}, T_{\max}]$ be denoted by T, i.e., $T = T_{\max} - T_{\min}$. Through additive temperature scaling, we may assume without loss of generality that the temperature range is $[0, T]$ rather than $[T_{\min}, T_{\max}]$, which we will do throughout this paper.

Let

$$\boldsymbol{s}_{t_0,t_1}(\boldsymbol{x}) = (s_1, s_2, \ldots, s_n)$$

denote the *temperature sequence* for the above-described channel when the input is the binary sequence $\boldsymbol{x} = (x_1, \ldots, x_n)$ and the temperature at the start of the transmission is the base temperature. Here, s_i denotes the channel temperature after transmitting x_i. It holds that

$$s_i = \begin{cases} s_{i-1} + t_1 & \text{if } x_i = 1 \\ \max\{0, s_{i-1} - t_0\} & \text{if } x_i = 0 \end{cases} \tag{2}$$

where $s_0 = 0$.

A binary sequence $\boldsymbol{x}$ of length n is called a (T, t_0, t_1) *thermal-aware sequence* (TA-sequence) if and only if all s_i in $\boldsymbol{s}_{t_0,t_1}(\boldsymbol{x})$ do not exceed the maximum temperature T, i.e.,

$$s_i \leq T \qquad \forall i = 1, 2, \ldots, n.$$

The set of all (T, t_0, t_1) TA-sequences of length n is denoted by $\mathcal{A}(T, t_0, t_1, n)$. The channel that only accepts (T, t_0, t_1) TA-sequences is called the (T, t_0, t_1) *thermal-aware channel* (TA-channel). The *capacity* of this channel is the maximum achievable asymptotic rate, i.e.,

$$\mathsf{cap}_{\mathsf{TA}}(T, t_0, t_1) = \limsup_{n\to\infty} \frac{\log_2 |\mathcal{A}(T, t_0, t_1, n)|}{n}. \tag{3}$$

Besides the additive scaling applied to have a base temperature of 0, we can also apply further multiplicative temperature scaling, for convenience in channel analysis. Note that

$$\mathcal{A}(\alpha T, \alpha t_0, \alpha t_1, n) = \mathcal{A}(T, t_0, t_1, n) \tag{4}$$

for any positive real number α. Hence, any $(\alpha T, \alpha t_0, \alpha t_1)$ TA-channel can be considered as being equivalent to the (T, t_0, t_1) TA-channel. The following two choices of α are of particular interest.

- By choosing $\alpha = 1/t_0$, we obtain a representation that is normalized to the cooling gradient t_0, i.e., the $(M, 1, k)$ TA-channel, where $M = T/t_0$ and k is given by (1).

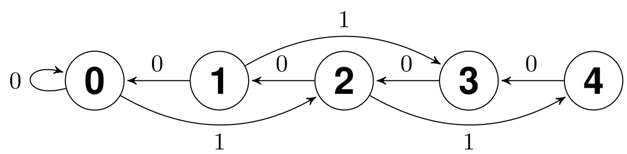

Fig. 1. Labelled graph for the $(4, 1, 2)$ TA-channel.

- By choosing $\alpha = q/t_0$, we obtain the $(qT/t_0, q, p)$ TA-channel, in which all temperatures are integer. This channel can also be denoted as an (N, q, p) TA-channel, where $N = \lfloor qT/t_0 \rfloor$.

Note that the temperature sequence $\boldsymbol{s}_{q,p}(\boldsymbol{x}) = (s_1, s_2, \ldots, s_n)$ of any (N, q, p) TA-sequence $\boldsymbol{x}$ satisfies

$$s_i \in \{0, 1, \ldots, N\} \qquad \forall i = 1, 2, \ldots, n.$$

Hence, the (N, q, p) TA-channel can be represented by a labelled directed graph with $N + 1$ nodes. Every node $j \in \{0, 1, \ldots, N\}$ represents a channel temperature. For each $i = 0, 1, \ldots, N - p$, there is an edge from state i to state $i + p$, corresponding to a signal 1 which increases the temperature by p. For each $i = q, q + 1, \ldots, N$, there is an edge from state i to state $i - q$, associated with a signal 0 which decreases the temperature by q. Finally, for each $i = 0, 1, \ldots, q-1$, there is an edge from state i to state 0, associated with a signal 0 which let the channel return to the base temperature.

The $(N+1) \times (N+1)$ *transition matrix* associated with the graph is denoted by $D_{N,q,p} = (d_{i,j})$. For all $0 \le i, j \le N$, the $d_{i,j}$ entry equals one when there is an edge from node i to node j, and zero otherwise, i.e,

$$d_{i,j} = \begin{cases} 1 & \text{if } i = 0, 1, \ldots, N-p \text{ and } j = i + p \\ 1 & \text{if } i = q, q+1, \ldots, N \text{ and } j = i - q, \\ 1 & \text{if } i = 0, 1, \ldots, q-1 \text{ and } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1: Consider the channel with maximum temperature $N = 4$, cooling gradient $q = 1$, and heating gradient $p = 2$. The graph for the $(4, 1, 2)$ TA-channel is shown in Figure 1, while the transition matrix is

$$D_{4,1,2} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{5}$$

For the input sequences $\boldsymbol{x} = (1, 1, 0, 1, 0, 0, 0)$ and $\boldsymbol{y} = (0, 1, 0, 0, 0, 1, 1)$ the temperature sequences are

$$\boldsymbol{s}_{1,2}(\boldsymbol{x}) = (2, 4, 3, 5, 4, 3, 2)$$

and

$$\boldsymbol{s}_{1,2}(\boldsymbol{y}) = (0, 2, 1, 0, 0, 2, 4),$$

respectively. Hence, $\boldsymbol{y}$ is a $(4, 1, 2)$ TA-sequence, but $\boldsymbol{x}$ is not, since the channel temperature would exceed $N = 4$ after the transmission of x_4.

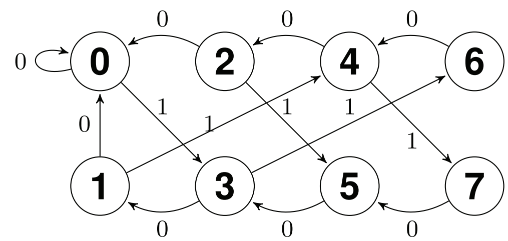

Fig. 2. Labelled graph for the $(7, 2, 3)$ TA-channel.

Example 2: Let $N = 7$, $q = 2$, and $p = 3$. The graph for the $(7, 2, 3)$ TA-channel is shown in Figure 2, while the transition matrix is

$$D_{7,2,3} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{6}$$

For the input sequences $\boldsymbol{x} = (1, 1, 0, 1, 0, 0, 0)$ and $\boldsymbol{y} = (0, 1, 0, 0, 0, 1, 1)$ the temperature sequences are

$$\boldsymbol{s}_{2,3}(\boldsymbol{x}) = (3, 6, 4, 7, 5, 3, 1)$$

and

$$\boldsymbol{s}_{2,3}(\boldsymbol{y}) = (0, 3, 1, 0, 0, 3, 6),$$

respectively. Hence, both $\boldsymbol{x}$ and $\boldsymbol{y}$ are $(7, 2, 3)$ TA-sequence, since all channel temperatures do not exceed $N = 7$.

Throughout this paper, we will use both the (N, q, p) and the $(M, 1, k)$ TA-channel/sequence representations, whichever is most appropriate. As mentioned, we can do this without loss of generality, since both are equivalent to the original (T, t_0, t_1) representation because of (4). In both cases, we should first measure t_0, t_1, T_{min}, and T_{max} to determine the parameters for the thermal-aware channel and use any one of them. Recall that the involved heating/cooling gradients are related through (1), while the maximum temperatures relate via

$$M = T/t_0 \text{ and } N = \lfloor qM \rfloor = \lfloor qT/t_0 \rfloor.$$

We will always assume $t_1 \le T$, and thus $k \le M$ and $p \le N$, since the all-zero sequence would be the only TA-sequence otherwise.

III. Properties of Thermal-Aware Sequences

In this section, several properties of TA-sequences will be presented. Most of these will be used in the next section to derive results on the thermal-aware channel capacity. A useful notation that will often be used is $\boldsymbol{x}[i, j]$ to indicate the substring $(x_i, x_{i+1}, \ldots, x_j)$ of a string $\boldsymbol{x}$.

Lemma 1: A binary sequence $\boldsymbol{x}$ is an $(M, 1, k)$ TA-sequence if and only if the weight of any substring $\boldsymbol{x}[i, j]$ of $\boldsymbol{x}$ is at most $(j - i + 1 + M)/(1 + k)$.

Proof: Consider the temperature sequence $\boldsymbol{s}_{1,k}(\boldsymbol{x}) = (s_1, s_2, \ldots, s_n)$ of $\boldsymbol{x}$. First, let $\boldsymbol{x}$ be an $(M, 1, k)$ TA-sequence. Suppose there exist i and j such that the substring $\boldsymbol{x}[i, j]$ of

$\boldsymbol{x}$ has weight w exceeding $(j-i+1+M)/(1+k)$. Then $s_j \geq s_{i-1} + wk - (j-i+1-w) \geq w(1+k) - (j-i+1) > M$, which contradicts $\boldsymbol{x}$ being an $(M,1,k)$ TA-sequence and this shows the "only if" statement in the lemma.

Next, let $\boldsymbol{x}$ be such that the weight of any substring $\boldsymbol{x}[i,j]$ of $\boldsymbol{x}$ is at most $(j-i+1+M)/(1+k)$. Suppose that $\boldsymbol{x}$ is not an $(M,1,k)$ TA-sequence. Then there exists j such that $s_j > M$. Let i be the largest index such that $s_{i-1} = 0$ and $i \leq j$. Then $M < s_j = wk - (j-i+1-w) = w(1+k) - (j-i+1)$, where w is the weight of $\boldsymbol{x}[i,j]$. We thus reach the contradiction that $w > (j-i+1+M)/(1+k)$ which shows the "if" statement in the lemma. ■

Because of this result, TA-sequences can also be characterized as sequences satisfying strong local weight constraints, and as such they could also be called *strongly locally bounded sequences*, in the same spirit as locally bounded sequences and strongly locally balanced sequences studied in [16].

The *running digital sum* (RDS) of a binary sequence $\boldsymbol{x}$ of length n is defined as $\boldsymbol{v} = \mathrm{RDS}(\boldsymbol{x}) = (v_1, v_2, \ldots, v_n)$ where $v_i = 2w(\boldsymbol{x}[1,i]) - i$, and $w(\boldsymbol{x}[1,i])$ is the Hamming weight of the substring $\boldsymbol{x}[1,i]$. Given a positive integer δ, a sequence $\boldsymbol{x}$ is said to be a δ-RDS-sequence if and only if $\max_i v_i - \min_i v_i \leqslant \delta$, where both the minimization and maximization are over all $0 \leq i \leq n$. For example, if $\boldsymbol{x} = (1,0,0,1,1,0,1,1)$ then $\boldsymbol{v} = (1,0,-1,0,1,0,1,2)$. Since $\max_i v_i = 2$ and $\min_i v_i = -1$, the sequence $\boldsymbol{x}$ is a 3-RDS-sequence. We note that δ-RDS-sequences have been well studied in Chapter 8 in [17].

Lemma 2: For all $k \leq 1$ and $M \geq k$, any $\lfloor M/k \rfloor$-RDS-sequence is an $(M,1,k)$ TA-sequence.

Proof: Let $\boldsymbol{x}$ be an $\lfloor M/k \rfloor$-RDS-sequence with $\mathrm{RDS}(\boldsymbol{x}) = (v_1, v_2, \ldots, v_n)$. Suppose that $\boldsymbol{x}$ is not an $(M,1,k)$ TA-sequence. According to Lemma 1 this implies that there exist i and j such that the weight w of the substring $\boldsymbol{x}[i,j]$ of $\boldsymbol{x}$ is exceeding $(j-i+1+M)/(1+k)$. Hence,

$$(j-i+1+M)/(1+k) < w \leq j-i+1$$

and thus

$$j-i+1 > M/k$$

$$\begin{aligned} v_j - v_{i-1} &= 2w - (j-i+1) \\ &> \frac{2(j-i+1+M)}{1+k} - (j-i+1) \\ &= (j-i+1)\frac{1-k}{1+k} + \frac{2M}{1+k} \\ &\geq \frac{M(1-k)}{k(1+k)} + \frac{2M}{1+k} = \frac{M}{k} \geq \left\lfloor \frac{M}{k} \right\rfloor \end{aligned}$$

which contradicts the fact that x is an $\lfloor M/k \rfloor$-RDS-sequence. Note that in the second inequality in this equation we use the condition that $k \leq 1$ as stated in the lemma. In conclusion, x is an $(M,1,k)$ TA-sequence. ■

For any positive integer d, the well-known *d-constrained sequences* are defined as binary sequences in which any two consecutive ones are separated by at least d zeroes [17]. For example, the sequence $(0,0,0,1,0,0,1,0,0,0,1,0,0)$ is a 2-constrained sequence as any two consecutive ones are separated by at least 2 zeroes.

Lemma 3: For all $M \geq k$, any $\lceil k \rceil$-constrained sequence is an $(M,1,k)$ TA-sequence.

Proof: Let $\boldsymbol{x}$ be a $\lceil k \rceil$-constrained sequence with temperature sequence $\boldsymbol{s}_{1,k}(\boldsymbol{x}) = (s_1, s_2, \ldots, s_n)$. Since there are at least $\lceil k \rceil$ zeroes in between any two subsequent ones in $\boldsymbol{x}$, it follows that $s_{i-1} = 0$ for every i for which $x_i = 1$, and thus $s_j \leq k \leq M$ for all j. We can thus conclude that $\boldsymbol{x}$ is an $(M,1,k)$ TA-sequence. ■

Lemma 4: For all $k \leq M < 2k$, any $(M,1,k)$ TA-sequence is a $\lceil 2k-M \rceil$-constrained sequence.

Proof: Let $\boldsymbol{x}$ be an $(M,1,k)$ TA-sequence with temperature sequence $\boldsymbol{s}_{1,k}(\boldsymbol{x}) = (s_1, s_2, \ldots, s_n)$. Suppose that there are two consecutive ones in $\boldsymbol{x}$ at positions i and $j > i$, such that the number of zeroes in between them is smaller than $\lceil 2k-M \rceil$. Then,

$$\begin{aligned} s_j \geq & s_{i-1} + k - (\lceil 2k-M \rceil - 1) + k \\ \geq & 2k + \lfloor M - 2k \rfloor + 1 = \lfloor M \rfloor + 1 > M, \end{aligned}$$

which contradicts that $\boldsymbol{x}$ is an $(M,1,k)$ TA-sequence. Hence, there are at least $\lceil 2k-M \rceil$ ones in between any two consecutive ones in $\boldsymbol{x}$. ■

Combining Lemmas 3 and 4, we obtain the following result.

Corollary 1: For all $k \leq M < 2k - \lceil k \rceil + 1$, a binary sequence is an $(M,1,k)$ TA-sequence if and only if it is a $\lceil k \rceil$-constrained sequence.

We note that $\lceil k \rceil$-constrained sequences are well studied in [17]. Hence, when the condition in Corollary 1 is satisfied, that is $k \leq M < 2k - \lceil k \rceil + 1$, we can use $(M,1,k)$ TA-sequences and $\lceil k \rceil$-constrained sequences alternatively as they are the same. In these cases, the thermal-aware channel capacity is the same as the one for the $\lceil k \rceil$-constrained channel which is known in [17]. In other cases, when the condition in Corollary 1 is not satisfied, the thermal-aware channel capacity is not known. We study the thermal-aware channel capacity in the following section.

IV. Thermal-Aware Channel Capacity

Since the TA-channel is a constrained channel which can be represented by a graph, it is known that the capacity of the constrained channel is the base-2 logarithm of the largest real eigenvalue of the graph's transition matrix [17], [18], [19]. Hence, for determining $\mathsf{cap}_{\mathsf{TA}}(N,q,p)$, we consider

$$\Gamma_{N,q,p}(z) = \det[zI - D_{N,q,p}]. \tag{7}$$

This determinant $\Gamma_{N,q,p}(z)$ is an $(N+1)$-th degree polynomial in z, and is called the *characteristic polynomial* of $D_{N,q,p}$. The capacity of the TA-channel

$$\mathsf{cap}_{\mathsf{TA}}(N,q,p) = \log_2 \lambda,$$

where λ is the largest real solution of the equation

$$\Gamma_{N,q,p}(z) = 0. \tag{8}$$

Several results for $\mathsf{cap}_{\mathsf{TA}}(M,1,k)$ are provided in Table I. As expected, we observe that $\mathsf{cap}_{\mathsf{TA}}(M,1,k)$ is increasing in M and decreasing in k.

Although the method presented in the previous paragraph applies to any set of channel parameters, there is still a need to have explicit expressions or estimates for the TA-channel

TABLE I

CAPACITY $\mathsf{cap}_{TA}(M, 1, k)$ FOR SELECTED VALUES OF M AND K

M	$k=1/2$	$k=2/3$	$k=1$	$k=4/3$	$k=3/2$	$k=2$	$k=5/2$	$k=3$	$k=7/2$	$k=4$	$k=5$
2	0.969757	0.925323	0.84955	0.679286	0.6509	0.551463	-	-	-	-	
3	0.990431	0.964523	0.91026	0.803641	0.759177	0.64060	0.52934	0.46496	-	-	-
4	0.996655	0.983464	0.94034	0.854675	0.81677	0.72631	0.597003	0.52448	0.45164	0.40569	-
5	0.998774	0.9911	0.95746	0.88822	0.859232	0.76745	0.662721	0.58152	0.498964	0.44894	0.36199
6	0.999541	0.995122	0.96812	0.911378	0.882798	0.79917	0.699842	0.63462	0.544883	0.49034	0.39519
7	0.999826	0.997202	0.97522	0.925765	0.899583	0.82017	0.730043	0.66284	0.588095	0.52910	0.42694
8	0.999934	0.998371	0.98019	0.936103	0.911923	0.83628	0.753438	0.68628	0.613699	0.56500	0.45681
9	0.999975	0.999035	0.98380	0.943895	0.92099	0.84836	0.769528	0.70509	0.635383	0.58535	0.48464
10	0.99999	0.999424	0.98650	0.949854	0.928043	0.85792	0.782481	0.71908	0.653458	0.60285	0.51046

capacity. For instance, note that if N is very large, it may be infeasible to determine the largest eigenvalue of $D_{N,q,p}$. Also, to analyze the behaviour of the capacity as a function of the channel parameters, such expressions may be useful. In the remainder of this section, we will present explicit expressions and bounds on the channel capacity for particular cases.

A. Bounds

In this subsection, we provide two explicit bounds based on results from the previous section in Theorems 1 and 2. In order to prove Theorem 1, we need the entropy function $H(x)$ defined by

$$H(x) = -x\log_2 x - (1) - x)\log_2(1-x),$$

where $0 < x < 1$. The following result was proved in [20] (Corollary 9 on page 310).

Lemma 5: Given $0 < \mu < 1/2$, the following equation holds.

$$\frac{2^{nH(\mu)}}{\sqrt{8n\mu(1-\mu)}} \leqslant \sum_{i=0}^{\mu n}\binom{n}{i} \leqslant 2^{nH(\mu)}.$$

Theorem 1: For any $M \geq k > 1$, we have that

$$\mathsf{cap}_{\mathrm{TA}}\,(M,1,k) \leqslant H(1/(1+k))$$

Proof: Let $\boldsymbol{x} = (x_1,\ldots,x_n)$ be an $(M,1,k)$-thermal-aware sequence of length n. According to Lemma 1 with $i = 1$ and $j = n$, the weight of $\boldsymbol{x}$ is at most $w^* = (n+M)/(1+k)$. Let $\mathcal{A}(n,w^*)$ be the set of all binary sequences of length n with weight at most w^*. Then,

$$\mathcal{A}(M,1,k,n) \subseteq \mathcal{A}\,(n,w^*)$$

Given M and k, we obtain $\lim_{n\to\infty} w^*/n = 1/(1+k)$. Hence, it follows from Lemma 5 that $\lim_{n\to\infty}\left(\log_2|\mathcal{A}\,(n,w^*)|\right)/n = H(1/(1+k))$. Therefore, the capacity of the $(M,1,k)$ TAchannel is upper bounded by $H(1/(1+k))$. ■

Note that this theorem implies that when extending Table I with larger values of M, the value of $\mathsf{cap}_{\mathrm{TA}}(M,1,k)$ would grow with M, but it does not exceed $H(1/(1+k))$ for any $M \geq k > 1$. For example, $\mathsf{cap}_{\mathrm{TA}}(M,1,2) \leqslant H(1/3) \approx 0.9183$ for any $M \geq 2$.

According to Lemma 2, every $\lfloor M/k \rfloor$-RDS-sequence is an $(M,1,k)$ TA-sequence if $k \leq 1$. The capacity of the δ-RDS-channel, which is the binary noiseless channel admitting only δ-RDS-sequences, has been studied in Chapter 8 in [17]. For any positive integer δ, the capacity of the channel is calculated on page 203 of [17] to be $\log_2(2\cos(\pi/(\delta+2)))$. Hence, $\mathsf{cap}_{\mathrm{TA}}(M,1,k)$ is lower bounded by $\log_2(2\cos(\pi/(\lfloor M/k\rfloor+2)))$. We state the result formally as follows.

Theorem 2: For any $k \leq 1$ and $M \geq k$, it holds that

$$\mathsf{cap}_{\mathrm{TA}}(M,1,k) \geqslant \log_2(2\cos(\pi/(\lfloor M/k\rfloor+2)))$$

Note that Theorem 2 implies that for any $k \leq 1$, the capacity $\mathsf{cap}_{\mathrm{TA}}(M,1,k)$ tends to 1 when M grows large.

B. The Case That k Is Integer

In this subsection, the focus is on the channel capacity for the case where k is an integer, that is, $q = 1$ and $p = k$. To this end, we investigate the characteristic polynomial $\Gamma_{N,1,p}(z)$ of the matrix $D_{N,1,p}$.

First, we define a slightly different matrix, namely the $(N+1)\times(N+1)$ matrix $E_{N,1,p}$, by

$$e_{i,j} = \begin{cases} 0 & \text{if} i = j = 0 \\ d_{i,j} & \text{otherwise,} \end{cases}$$

where the $d_{i,j}$ are the entries from $D_{N,1,p}$. Let

$$\Phi_{N,1,p}(z) = \det[zI - E_{N,1,p}] \tag{9}$$

denote the characteristic polynomial of $E_{N,1,p}$. We observe that, for the case $p = 1$, $E_{N,1,p}$ is the transition matrix of the state machine for N-RDS codes, which have been well studied in [17] [Chapter 8]. The first few characteristic polynomials $\Phi_{N,1,1}(z)$ can easily be evaluated by hand:

$$\Phi_{1,1,1}(z) = z^2 - 1,$$
$$\Phi_{2,1,1}(z) = z^3 - 2z,$$
$$\Phi_{3,1,1}(z) = z^4 - 3z^2 + 1.$$

This sequence of polynomials is known under the name *Vieta-Fibonacci* (VF). The VF-polynomials $V_n(z)$ have been originally defined by the recursion $V_n(z) = zV_{n-1}(z) - V_{n-2}(z)$, with $V_0(z) = 0$ and $V_1(z) = 1$. These are closely related to the well-known Chebyshev polynomials of the second kind [21]. It holds that

$$\Phi_{N,1,1}(z) = V_{N+2}(z) = \frac{\sin((N+2)\arccos(z/2))}{\sin(\arccos(z/2))}. \tag{10}$$

An explicit expression for VF-polynomials derived in [21] is

$$V_n(z) = \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^i\binom{n-1-i}{i}z^{n-1-2i},$$

which implies

$$\Phi_{N,1,1}(z) = \sum_{i=0}^{\lfloor \frac{N+1}{2} \rfloor} (-1)^i \binom{N+1-i}{i} z^{N+1-2i}.$$

For $1 \leq p \leq N$, from Equation (9), we can obtain the recursion $\Phi_{N,1,p}(z) = z\Phi_{N-1,1,p}(z) - \Phi_{N-p-1,1,p}(z)$. Following the proof in [21] to find the explicit expression for VF-polynomials, we obtain

$$\Phi_{N,1,p}(z) = \sum_{i=0}^{\lfloor \frac{N+1}{p+1} \rfloor} (-1)^i \binom{N+1-pi}{i} z^{N+1-i(p+1)}. \tag{11}$$

For example, this gives $\Phi_{N,1,N}(z) = z^{N+1} - 1$, which is indeed equal to the determinant from (9) with $p = N$.

For the characteristic polynomial $\Gamma_{N,1,p}(z)$, we can easily derive an expression for the case $p = N$ by first-column expansion of the determinant from (7):

$$\Gamma_{N,1,N}(z) = z^{N+1} - z^N - 1. \tag{12}$$

For the cases that $p < N$, we can make use of a connection between $\Phi_{N,1,p}(z)$ and $\Gamma_{N,1,p}(z)$, which follows from the fact that the matrices $zI - D_{N,1,p}$ and $zI - E_{N,1,p}$ are identical, except for the upper-left entries, which equal $z-1$ and z, respectively. It immediately follows that

$$\Gamma_{N,1,p}(z) = \Phi_{N,1,p}(z) - \Phi_{N-1,1,p}(z) \tag{13}$$

for any $N \geq 2$ and $1 \leq p < N$. We see $\Phi_{N,1,p}(z)$ and $\Gamma_{N,1,p}(z)$ are closely related, or in other words, the RDS sequences and the thermal-aware sequences with integer k are closely related. Using Equations (11) and (13), we can find the characteristic polynomials $\Gamma_{N,1,p}(z)$ and compute the capacity of the channel. For example, the first few characteristic polynomials are

$$\begin{aligned}
\Gamma_{2,1,1}(z) &= \Phi_{2,1,1}(z) - \Phi_{1,1,1}(z) \\
&= z^3 - z^2 - 2z + 1, \\
\Gamma_{3,1,1}(z) &= \Phi_{3,1,1}(z) - \Phi_{2,1,1}(z) \\
&= z^4 - z^3 - 3z^2 + 2z + 1, \\
\Gamma_{3,1,2}(z) &= \Phi_{3,1,2}(z) - \Phi_{2,1,2}(z) \\
&= z^4 - z^3 - 2z + 1.
\end{aligned}$$

By finding the largest real roots of these polynomials we can obtain the capacity of the corresponding TA-channels. For $p = q = 1$, i.e., the case that the heating and cooling gradients are equal, we derive an explicit formula for the capacity of the (N, q, p) thermal-aware channel as follows.

Theorem 3: The capacity of the $(N, 1, 1)$ TA-channel is

$$\mathsf{cap}_{\mathsf{TA}}(N, 1, 1) = \log_2(2\cos(\pi/(2N+3))).$$

Proof: We have

$$\begin{aligned}
\Gamma_{N,1,1}(z) &= \Phi_{N,1,1}(z) - \Phi_{N-1,1,1}(z) \\
&= \frac{\sin((N+2)x)}{\sin x} - \frac{\sin((N+1)x)}{\sin x} \\
&= 2\cos\left(\frac{2N+3}{2}x\right)\frac{\sin(x/2)}{\sin x},
\end{aligned}$$

where the first equality follows from (13) and the second from (10) using the notation $x = \arccos(z/2)$. Note that the displayed expression equals zero if and only if $x = (2i+1)\pi/(2N+3)$, $i = 0, 1, \ldots, N$. Hence, the eigenvalues of $\Gamma_{N,1,1}(z)$ are $2\cos((2i+1)\pi/(2N+3))$, from which it follows that the largest eigenvalue equals $2\cos(\pi/(2N+3))$. In conclusion, the capacity of the $(N, 1, 1)$ TA-channel equals $\log_2(2\cos(\pi/(2N+3)))$. ■

C. Approximations

As discussed, the capacity of a TA-channel can be determined via the eigenvalues of its transition matrix. Since this matrix is of size $(N+1) \times (N+1)$, where $N = \lfloor qM \rfloor$, this approach may be infeasible when q is large. However, since $\mathsf{cap}_{\mathsf{TA}}(M, 1, k)$ is non-increasing in k, it can be approximated as follows for large values of q (and p). Let p', q', p'', and q'' be positive integers such that

$$\frac{p'}{q'} \approx k = \frac{p}{q} \approx \frac{p''}{q''} \text{ and } \frac{p'}{q'} \leq k = \frac{p}{q} \leq \frac{p''}{q''},$$

where both q' and q'' are considerably smaller than q. Then

$$\mathsf{cap}_{\mathsf{TA}}(M, 1, p''/q'') \leq \mathsf{cap}_{\mathsf{TA}}(M, 1, k) \leq \mathsf{cap}_{\mathsf{TA}}(M, 1, p'/q'). \tag{14}$$

If it is feasible to calculate the first and the third capacities in (14) and if these are close to each other, then we obtain a good approximation for $\mathsf{cap}_{\mathsf{TA}}(M, 1, k)$. For example, when $M = 10$, for all $1/2 \leqslant k \leqslant 2/3$, we can estimate $0.999424 \leqslant \mathsf{cap}_{\mathsf{TA}}(M, 1, k) \leqslant 0.99999$ and the gap between upper bound and lower bound is less than 0.001. We note that there is a trade-off between the gap in (14) and the complexity. However, with the power of the computer nowadays, from practical point of view, we believe that it is possible to estimate the capacity as close as it is required. From theoretical point of view, the exact value and the behavior of the capacity when all parameters are large are of our interest.

Another way to approximate the capacity of the TA channel is to determine $\mathcal{A}(N, q, p, n)$ for sufficiently large n, and then approximate $\mathsf{cap}_{\mathsf{TA}}(N, q, p)$ by $(\log_2 |\mathcal{A}(N, q, p, n)|)/n$. Note that $\mathcal{A}(N, q, p, n)$ equals the sum of all entries in the top row of the matrix $D^n_{N,q,p}$. Since the size of the matrix $D_{N,q,p}$ is constant, it is a folklore that we can compute the matrix $D^n_{N,q,p}$ in $O(\log n)$ time for fixed N, q, and p using a general method known as exponentiation by squaring. In particular, we can compute all the matrices $D^{2^i}_{N,q,p}$ for $i = 1, 2, \ldots, \lfloor \log n \rfloor$ in $O(\log n)$ times. Since $D^n_{N,q,p}$ is a sum of at most $\lfloor \log n \rfloor$ matrices of fixed size, we can compute $D^n_{N,q,p}$ in $O(\log n)$ times. Thus, we can count the number of (N, q, p) TA-sequences of length n in $O(\log n)$ time for fixed N, q, and p.

V. Thermal-Aware Coding

After having studied the maximum asymptotic rate that can be achieved for communication over a TA-channel in the previous section, we will now focus on the maximum achievable rate when using TA-sequences of a finite length n. In principle, this would be $(\log_2 |\mathcal{A}(N, q, p, n)|)/n$, but if we require that such sequences should be freely cascadable without violating the TA-constraint, then it may be lower. Note that in Example 2 the sequences $\boldsymbol{y}$ and $\boldsymbol{x}$ are both $(7, 2, 3)$ TA-sequences, but that the cascaded sequence $(\boldsymbol{y}, \boldsymbol{x})$ is not.

Hence, $\boldsymbol{x}$ and $\boldsymbol{y}$ cannot both be included in a set of $(7, 2, 3)$ TA-sequences that is used to represent messages that are to be serially transmitted over the $(7, 2, 3)$ TA-channel.

For practical purposes, there is thus a desire for subsets $\mathcal{C}$ of $\mathcal{A}(N, q, p, n)$, called *codes*, such that sequences from $\mathcal{C}$ can be cascaded without violating the maximum temperature constraint N. Messages can be mapped to unique code sequences, leading to a rate $(\log_2 |\mathcal{C}|)/n$. If the messages are binary sequences of fixed length, then the maximum message length that can be accommodated by the code $\mathcal{C}$ is $\lfloor \log_2 |\mathcal{C}| \rfloor$, leading to a rate of $(\lfloor \log_2 |\mathcal{C}| \rfloor)/n$.

There are mature methods for designing codes based on finite-state machines, such as Franaszek's *successive elimination method* [22] and the state-splitting ACH method [23]. In this section, we will explore code design for the (N, q, p) TA-channel under consideration.

A. Preliminaries

For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$, we recursively define $\boldsymbol{v}_{q,p}(\boldsymbol{x}, u) = (v_1, v_2, \ldots, v_n)$, $u \in \{0, 1, \ldots, N\}$, by $v_0 = u$ and

$$v_i = \begin{cases} v_{i-1} + p & \text{if} x_i = 1, \\ \max\{0, v_{i-1} - q\} & \text{if} x_i = 0, \end{cases} \tag{15}$$

for $i = 1, 2, \ldots, n$. Note that $\boldsymbol{s}_{q,p}(\boldsymbol{x})$ considered earlier is equal to $\boldsymbol{v}_{q,p}(\boldsymbol{x}, 0)$.

Let $\mathcal{S}_{u,v}$, with $u, v \in \{0, 1, \ldots, N\}$, denote the subset of $\{0, 1\}^n$ which contains all (N, q, p) TA-sequences $\boldsymbol{x}$ emerging from state u and ending in state v, i.e., for which $\boldsymbol{v}_{q,p}(\boldsymbol{x}, u)$ satisfies $v_i \le N$ for all i and $v_n = v$.

The encoder has a set of states, denoted by Σ^*, which is assumed to be a subset of the set $\Sigma = \{0, 1, \ldots, N\}$ of nodes in the graph. The set of valid TA-sequences that emerge from $u \in \Sigma^*$ and terminate in one of the states in Σ^*, denoted by $\mathcal{F}_{u,\Sigma^*}$, is defined by

$$\mathcal{F}_{u,\Sigma^*} = \bigcup_{v \in \Sigma^*} \mathcal{S}_{u,v}. \tag{16}$$

We have

$$|\mathcal{F}_{u,\Sigma^*}| = \sum_{v \in \Sigma^*} d_{u,v}^{[n]}, \tag{17}$$

where $d_{u,v}^{[n]}$ denotes the (u, v)-th element of $D_{N,q,p}^n$. Any Σ^* satisfying

$$\min_{u \in \Sigma^*} \sum_{v \in \Sigma^*} d_{u,v}^{[n]} \geqslant |\mathcal{M}| \tag{18}$$

is an acceptable encoder state set for encoding a message set $\mathcal{M}$. Such a state set can be found by exhaustive search of all possible subsets of Σ or by application of Franaszek's method [22].

A decoder requires, in general, the knowledge of the encoder state, which may result in additional errors when encoded words are received in error. Note that in the case at hand the decoder cannot, in general, uniquely restore the channel state by observing a limited number of consecutive symbols. *State-independent decoding* is attractive for circumventing such error propagation.

B. State-Independent Encoding and Decoding

Let

$$\mathcal{S}_{\Sigma^*} = \bigcap_{u \in \Sigma^*} \mathcal{F}_{u,\Sigma^*} \tag{19}$$

denote the intersection of the $|\Sigma^*|$ sets $\mathcal{F}_{u,\Sigma^*}$. The set $\mathcal{S}_{\Sigma^*}$ consists of admissible binary sequences that can start and terminate in any of the states in Σ^*. A single look-up table for encoding and decoding is possible if

$$|\mathcal{S}_{\Sigma^*}| \geqslant |\mathcal{M}|. \tag{20}$$

If (20) is satisfied, a subset of the words in $\mathcal{S}_{\Sigma^*}$ can be uniquely assigned (paired) to the $|\mathcal{M}|$ source words, thus constructing a one-to-one relationship between source and codewords.

In the above, we have tacitly assumed that we conduct an exhaustive search for an acceptable state set Σ^*. Freiman and Wyner [24] showed that it is not necessary to consider all possible state subsets Σ^*. It is sufficient to consider particular state subsets, called *complete terminal sets*. Define the partial ordering '$\prec$' on the states: that is, $u \prec v$, if *every* n-bit sequence admissible from state u is also admissible from state v, in other words, if $\cup_{j \in \Sigma} S_{u,j} \subseteq \cup_{j \in \Sigma} S_{v,j}$. A state set Σ' is a complete terminal set if $u \in \Sigma'$, $u \prec v \Rightarrow v \in \Sigma'$. For the model under consideration in this paper, this implies that we only need to consider state subsets of the form $\{0, 1, \ldots, a\}$, with $a \in \{0, 1, \ldots, N\}$, as shown next.

Lemma 6: Let $\boldsymbol{x}$ be an admissible sequence from state u and ending in state w, with $1 \leqslant u \leqslant N$ and $0 \leqslant w \leqslant N$. Then $\boldsymbol{x}$ is also an admissible sequence from state $u - 1$, ending in state $w' \leqslant w$.

Proof: Denote $\boldsymbol{v}_{q,p}(\boldsymbol{x}, u)$ by $(v_1, v_2, \ldots, v_n)$ and $\boldsymbol{v}_{q,p}(\boldsymbol{x}, u - 1)$ by $(v_1', v_2', \ldots, v_n')$. Let i^* be the smallest value of $i \in \{1, 2, \ldots, n\}$ such that $v_{i-1} \le q$ and $x_i = 0$. Set $i^* = n + 1$ if such an i does not exist. Then,

$$v_i' = \begin{cases} v_i - 1 & \text{if } i < i^* \\ v_i & \text{if } i \geqslant i^* \end{cases} \tag{21}$$

Since $v_i \leqslant N$, also $v_i' \leqslant N$ for all $1 \leqslant i \leqslant n$, and so $\boldsymbol{x}$ is admissible from state $u - 1$, ending in state $w' = v_n' \leqslant v_n = w$. ■

From Lemma 6 and the definition of $\prec$ we have the following result.

Corollary 2: For any state u with $1 \leqslant u \leqslant N$, $u \prec u - 1$. Hence, any complete terminal set is of the form $\{0, 1, \ldots, a\}$, with $1 \leqslant a \leqslant N$. Furthermore, from Lemma 6 and the definition of $\mathcal{F}_{u,\Sigma^*}$, we also have the following result.

Corollary 3: For any state set $\Sigma^* = \{0, 1, \ldots, a\}$ with $0 \leqslant a \leqslant N$, it holds that $\mathcal{F}_{u,\Sigma^*} \subseteq \mathcal{F}_{u-1,\Sigma^*}$, for all $1 \leqslant u \leqslant a$.

Hence, for all $0 \leqslant a \leqslant N$, it holds that

$$\begin{aligned} S_{\{0,1,\ldots,a\}} &= \bigcap_{u \in \{0,1,\ldots,a\}} \mathcal{F}_{u,\{0,1,\ldots,a\}} \\ &= \mathcal{F}_{a,\{0,1,\ldots,a\}}. \end{aligned}$$

For clerical convenience, let $\mathcal{F}_{a,\{0,1,\ldots,a\}}$ be denoted by $\mathcal{C}_a$. Hence, $\mathcal{C}_a$ is the subset of $\{0, 1\}^n$ which contains all vectors $\boldsymbol{x}$ for which $\boldsymbol{v}_{q,p}(\boldsymbol{x}, a)$ satisfies $v_i \le N$ for all i and $v_n \le a$.

Let $m_a = \lfloor \log_2 |\mathcal{C}_a| \rfloor$. The following results follow from the above analysis.

Theorem 4: For any $0 \leqslant a \leqslant N$, the code $\mathcal{C}_a \subseteq \mathcal{A}(N, q, p, n)$ can be used to encode (decode) messages from a message set of size at most $|\mathcal{C}_a|$ into (from) binary codewords of length n. Cascading such codewords results in a valid (N, q, p) TA-sequence.

Corollary 4: For any $0 \leqslant a \leqslant N$, there exists a subset of $\mathcal{C}_a$ that can act as a code to encode (decode) binary messages of length m_a into (from) binary codewords of length n. Cascading such codewords results in a valid (N, q, p) TA-sequence.

Example 3: For $N = 3$, $q = 1$, $p = 2$, and $n = 12$, we obtain

$$D^{12}_{3,1,2} = \begin{bmatrix} 98 & 56 & 83 & 36 \\ 83 & 51 & 76 & 27 \\ 56 & 27 & 51 & 20 \\ 36 & 20 & 27 & 15 \end{bmatrix}. \quad (22)$$

Hence, $|\mathcal{C}_0| = 98$, $|\mathcal{C}_1| = 83 + 51 = 134$, $|\mathcal{C}_2| = 56 + 27 + 51 = 134$, and $|\mathcal{C}_3| = 36 + 20 + 27 + 15 = 98$. Thus, $|\mathcal{C}_1|$ has a rate $\log_2(134)/12 \approx 0.589$, which is about 92% of the capacity of the $(3, 1, 2)$ TA-channel. Further, we have $m_0 = m_3 = \lfloor \log_2(98) \rfloor = 6$ and $m_1 = m_2 = \lfloor \log_2(134) \rfloor = 7$. Thus, there exists a code encoding binary messages of length 7 into codewords of length 12. The achieved code rate $7/12$ is about 91% of capacity.

C. Optimal Codes

For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$, let the reversed vector be denoted by $\boldsymbol{x}^R$, i.e.,

$$\boldsymbol{x}^R = (x_n, x_{n-1}, \ldots, x_1).$$

Let $\boldsymbol{t}_{q,p}(\boldsymbol{x}) = (t_1, t_2, \ldots, t_n)$ be defined by

$$t_i = \sum_{j=1}^{i} (px_j + q(x_j - 1)) = -qi + (p + q) \sum_{j=1}^{i} x_j,$$

$i = 0, 1, \ldots, n$. Note that $\boldsymbol{t}_{q,p}(\boldsymbol{x})$ is a kind of "weighted" running digital sum of $\boldsymbol{x}$, where each zero contributes $-q$ and each one contributes p, with t_i starting the count at index 1 and stopping at index i.

In order to optimize the code rate, it follows from the previous subsection that we should determine for which value of $a \in \{0, 1, \ldots, N\}$ the size of $\mathcal{C}_a$ is the largest. In this subsection, we will show that the maximum is established for $a = \lfloor N/2 \rfloor$ if $q = 1$, i.e., if k is an integer.

Theorem 5: For all $a \in \{0, 1, \ldots, N\}$ it holds that

$$x \in \mathcal{C}_a \Leftrightarrow x^R \in \mathcal{C}_{N-a}.$$

Proof: Let $\boldsymbol{x}$ be any sequence in $\mathcal{C}_a$, and let $\boldsymbol{v}_{q,p}(\boldsymbol{x}, a) = (v_1, v_2, \ldots, v_n)$, $\boldsymbol{t}_{q,p}(\boldsymbol{x}) = (t_1, t_2, \ldots, t_n)$, $\boldsymbol{v}_{q,p}(\boldsymbol{x}^R, N - a) = (v'_1, v'_2, \ldots, v'_n)$, and $\boldsymbol{t}_{q,p}(\boldsymbol{x}^R) = (t'_1, t'_2, \ldots, t'_n)$. We first show that $\boldsymbol{x} \in \mathcal{C}_a$ implies $\boldsymbol{x}^R \in \mathcal{C}_{N-a}$, i.e., a) $v'_i \leq N$ for all i and b) $v'_n \leq N - a$.

a) In order to prove that $v'_i \leq N$ for all i, we suppose that there exists a $j \in \{1, 2, \ldots, n\}$ such that $v'_j > N$ and show that this contradicts the fact that $\boldsymbol{x} \in \mathcal{C}_a$. To this end, we consider for such a j two cases: (i) there is no $g \leq j$ such that $v'_g = 0$, and (ii) there does exist such a g. In case (i), it holds that $v'_j = N - a + t'_j$ and thus

$$t'_j = v'_j - N + a > N - N + a = a$$

Hence,

$$\begin{aligned} v_n &\geq v_{n-j} + t_n - t_{n-j} \\ &= v_{n-j} + t'_j > 0 + a = a, \end{aligned}$$

which contradicts $\boldsymbol{x} \in \mathcal{C}_a$. In case (ii), let g^* denote the largest g such that $v'_g = 0$ and $g \leq j$. Note that

$$t'_j - t'_{g^*} = v'_j - v'_{g^*} > N - 0 = N$$

This implies

$$\begin{aligned} v_{n-g^*} &\geq v_{n-j} + t_{n-g^*} - t_{n-j} \geq t_{n-g^*} - t_{n-j} \\ &= t'_j - t'_{g^*} > N, \end{aligned}$$

which contradicts $\boldsymbol{x} \in \mathcal{C}_a$. In conclusion, $v'_i \leq N$ for all i.

b) In order to prove that $v'_n \leq N - a$, we show that this inequality not being true contradicts the fact that $\boldsymbol{x} \in \mathcal{C}_a$. So suppose that $v'_n > N - a$. We again consider two cases: (i) there is no $g \leq n$ such that $v'_g = 0$, and (ii) there does exist such a g. In case (i), it holds that $v'_n = N - a + t'_n$ and thus

$$t'_n = v'_n - N + a > N - a - N + a = 0.$$

Hence,

$$v_n \geq a + t_n = a + t'_n > a$$

which contradicts $x \in \mathcal{C}_a$. In case (ii), let g^* denote the largest g such that $v'_g = 0$ and $g \leq n$. Note that

$$t'_n - t'_{g^*} = v'_n - v'_{g^*} = v'_n > N - a$$

This implies

$$v_{n-g^*} \geq a + t_{n-g^*} = a + t'_n - t'_{g^*} > a + N - a = N$$

which contradicts $\boldsymbol{x} \in \mathcal{C}_a$. In conclusion, $v'_n \leq N - a$.

It follows from a) and b) that the "$\Rightarrow$" statement holds, and thus the "$\Leftarrow$" statement holds as well, since $(\boldsymbol{x}^R)^R = \boldsymbol{x}$. ■

Corollary 5: It holds for all $a \in \{0, 1, \ldots, N\}$ that

$$|\mathcal{C}_a| = |\mathcal{C}_{N-a}|$$

Theorem 6: If $q = 1$ and $0 \leq a \leq N/2 - 1$, then

$$|\mathcal{C}_a| \leq \left|\mathcal{C}_{a+1}\right|$$

Proof: The proof is established by providing an injective mapping f from $\mathcal{C}_a$ to $\mathcal{C}_{a+1}$. Let $\boldsymbol{x}$ be any sequence in $\mathcal{C}_a$, and let $\boldsymbol{v}_{q,p}(\boldsymbol{x}, a) = (v_1, v_2, \ldots, v_n)$, $\boldsymbol{t}_{q,p}(\boldsymbol{x}) = (t_1, t_2, \ldots, t_n)$, and $\boldsymbol{t}_{q,p}(\boldsymbol{x}^R) = (t'_1, t'_2, \ldots, t'_n)$. Define $z_{\boldsymbol{x}}$ as the smallest index for which the running digital sum of $\boldsymbol{x}$ becomes negative, i.e.,

$$t_i \geq 0 \ \ \forall 1 \leq i \leq z_{\boldsymbol{x}} - 1 \text{ and } t_i = -1 \text{ for} i = z_{\boldsymbol{x}}.$$

If such an index does not exist, i.e., if $t_i \geq 0 \ \ \forall 1 \leq i \leq n$, then set $z_{\boldsymbol{x}} = n$.

We decompose $\boldsymbol{x}$ as

$$x = (u, w)$$

with $\boldsymbol{u}$ being of length $n - z$ and $\boldsymbol{w}$ being of length z, where $z = z_{\boldsymbol{x}^R}$, and map $\boldsymbol{x}$ to

$$y = f(x) = \left(w^R, u\right)$$

Let $\boldsymbol{v}_{q,p}(\boldsymbol{y}, a + 1) = (v''_1, v''_2, \ldots, v''_n)$ and $\boldsymbol{t}_{q,p}(\boldsymbol{y}) = (t''_1, t''_2, \ldots, t''_n)$. We will show that $\boldsymbol{y}$ is in $\mathcal{C}_{a+1}$, that is, it holds that a) $v_i'' \le N$ for all i and b) $v_n'' \le a+1$, and, furthermore, that $\boldsymbol{y}$ is unique for every $\boldsymbol{x}$. Observe that if there exists an index i such that $t_i' = -1$, then we have from the definitions of z and $\boldsymbol{y}$ that

$$v_z'' = a + 1 + t_z' = a + 1 - 1 = a$$

and

$$v_i'' = v_{i-z} \tag{23}$$

for all $z+1 \le i \le n$.

a) Note that $t_i'' = t_i' \le a$ for all $1 \le i \le z$, since $t_j' \ge a+1$ for any $j \in \{1, 2, \ldots, z\}$ would imply that $v_n \ge v_{n-j} + t_j' \ge a+1$, which contradicts $\boldsymbol{x} \in \mathcal{C}_a$. Hence,

$$v_i'' = a + 1 + t_i'' \le a + 1 + a = 2a + 1 \le N$$

for all $1 \le i \le z$. If $z < n$, then we have from (23) that

$$v_i'' = v_{i-z} \le N$$

for all $z+1 \le i \le n$.

b) If there exists an index i such that $t_i' = -1$, then we have from (23) that

$$v_n'' = v_{n-z} \le v_n - t_z' \le a + 1$$

where the latter inequality follows from the facts that $v_n \le a$ and $t_z' = -1$. If there does not exist an index i such that $t_i' = -1$, then $z = n$, $\boldsymbol{y} = \boldsymbol{x}^R$, and thus

$$v_n'' = a + 1 + t_n' = a + 1 + t_n \le a + 1$$

The inequality follows from the fact that $t_n \le 0$, since $t_n > 0$ would imply $v_n \ge a + t_n > a$, which contradicts $\boldsymbol{x} \in \mathcal{C}_a$.

We conclude that a) $v_i'' \le N$ for all i and b) $v_n'' \le a+1$, and thus that $\boldsymbol{y} \in \mathcal{C}_{a+1}$. Finally, note that $z = z_{\boldsymbol{x}^R} = z_{\boldsymbol{y}}$, so we can retrieve z from $\boldsymbol{y}$ and thus establish the inverse mapping $f^{-1}(\boldsymbol{y}) = f^{-1}((\boldsymbol{w}^R, \boldsymbol{u})) = (\boldsymbol{u}, (\boldsymbol{w}^R)^R) = (\boldsymbol{u}, \boldsymbol{w}) = \boldsymbol{x}$. Hence, f is indeed an injective mapping from $\mathcal{C}_a$ to $\mathcal{C}_{a+1}$, which proves the statement in the theorem.■

Combining Corollary 5 and Theorem 6 gives us the following important result.

Corollary 6: If $q = 1$, then the cardinality of $\mathcal{C}_a$ is maximized for $a = \lfloor N/2 \rfloor$, i.e.,

$$|\mathcal{C}_a| \le \left|\mathcal{C}_{\lfloor N/2 \rfloor}\right| \text{ for all } a = 0, 1, \ldots, N.$$

While this result solves the problem of determining the largest $\mathcal{C}_a$ in the case that $q = 1$, the problem is still open in the case that $q > 1$. One could conjecture that $\mathcal{C}_{\lfloor N/2 \rfloor}$ is optimal for any q, but this is not true. The next example shows a case in which $|\mathcal{C}_a|$ is not maximized for $a = \lfloor N/2 \rfloor$.

Example 4: Let $N = 7$, $q = 3$, $p = 4$, and $n = 5$. Then

$$D_{7,3,4}^5 = \begin{bmatrix} 5 & 2 & 1 & 0 & 3 & 1 & 1 & 0 \\ 5 & 2 & 1 & 0 & 3 & 1 & 0 & 1 \\ 4 & 3 & 1 & 0 & 3 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Observe that $|\mathcal{C}_0| = |\mathcal{C}_7| = 5$, $|\mathcal{C}_1| = |\mathcal{C}_6| = 7$, $|\mathcal{C}_2| = |\mathcal{C}_5| = 8$, and $|\mathcal{C}_3| = |\mathcal{C}_4| = 7$. Hence $|\mathcal{C}_a|$ is maximized for $a = 2$ and $a = 5$.

D. Generating Functions

Generating functions [25] are a very useful tool for enumerating codewords. Define the generating function

$$G(z) = \sum g_i z^i. \tag{24}$$

The operation $[z^n]G(z)$ denotes the *extraction* of the coefficient of z^n in the formal power series $G(z)$, that is,

$$[z^n]G(z) = [z^n]\left(\sum g_i z^i\right) = g_n. \tag{25}$$

For the model under consideration, let the elements, $h_{i,j}(z)$, of the $(N+1) \times (N+1)$ matrix $H(z)$ denote the generating functions of the number of valid binary sequences starting in state i and ending in state j. Invoking the *transfer-matrix method* [26], we yield

$$H(z) = (I - zD_{N,q,p})^{-1}, \tag{26}$$

where

$$h_{i,j}(z) = \frac{\Delta_{j,i}}{\Delta}, \tag{27}$$

$$\Delta = \det[I - zD_{N,q,p}] = z^{N+1}\Gamma_{N,q,p}(1/z), \tag{28}$$

and $\Delta_{j,i}$ is the j,i-th cofactor (signed minor) of $[I - zD_{N,q,p}]$. From the analysis in the previous subsection it follows that the optimal code size for a code of length n equals

$$M_{N,q,p}(n) = [z^n] \sum_{j=0}^{\lfloor N/2 \rfloor} h_{\lfloor N/2 \rfloor, j}(z) \tag{29}$$

if $q = 1$. Below we work out a special case.

Example 5: We consider the same setting as in Example 3, i.e., $N = 3$, $q = 1$, and $p = 2$. For this case we have $\{0, 1\}$ as an optimal encoder state set. From (26)–(29) it follows that the maximum code size is

$$M_{3,1,2}(n) = [z^n] \sum_{j=0}^{1} h_{1,j}(z) = [z^n] \frac{1}{z^4 - 2z^3 - z + 1}. \tag{30}$$

This implies the recursion relation for $n > 0$,

$$M_{3,1,2}(n) = M_{3,1,2}(n-1) + 2M_{3,1,2}(n-3) - M_{3,1,2}(n-4), \tag{31}$$

where $M_{3,1,2}(i) = 0$, $i < 0$, and $M_{3,1,2}(0) = 1$. After an evaluation, we obtain

$$\begin{aligned} M_{3,1,2}(n) = [z^n](&1 + z + z^2 + 3z^3 + 4z^4 \\ &+ 5z^5 + 10z^6 + 15z^7 + 21z^8 + 36z^9 \\ &+ 56z^{10} + 83z^{11} + 134z^{12} + \cdots). \end{aligned} \tag{32}$$

This integer sequence is related to sequences A176848, A052916, A113435 in [27] as all these four sequences have the same linear recurrence with the constant coefficients (1,0,2,-1). They differ in their initial conditions. Note that the coefficient of z^{12} in (32) equals 134, which is in agreement with Example 3.

VI. Conclusion and Future Work

We have described a thermal-aware channel model and thoroughly analyzed its capacity. Furthermore, fixed-length codes for such channels have been presented. Maximum-sized codes have been identified for the case that the ratio of the heating and cooling gradients is an integer. The non-integer case is still open.

An interesting question is to enumerate exactly the number of all (N, p, q) TA-sequences of length n, given four integers N, p, q, n. Although there is an algorithm to determine $\mathcal{A}(N, q, p, n)$ as mentioned in Subsection IV-C, it is only efficient when N, q, p are constant. Hence, we are interested in an explicit formula for $\mathcal{A}(N, q, p, n)$ for all parameters. We observe that an (N, p, q) TA-sequence is closely related to a generalized Dyck path of bounded height [28]. In our future work, we will exploit some enumerative techniques to find an explicit formula for $\mathcal{A}(N, q, p, n)$.

Another interesting question is the behaviour of thermal-aware communication using multiple wires [29]. Multiple wires can be used to transmit codewords to a receiver. The combination of the properties of one wire used in a multiple wires channel to send information is intriguing. This extension is left for future work as it is beyond the scope of this paper.

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