Importance of Symbol Equity in Coded Modulation for Power Line Communications

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***Abstract*—The use of multiple frequency shift keying modulation with permutation codes addresses the problem of permanent narrowband noise disturbance in a power line communications system. In this paper, we extend this coded modulation scheme based on permutation codes to general codes and introduce an additional new parameter that more precisely captures a code's performance against permanent narrowband noise. As a result, we define a new class of codes, namely, equitable symbol weight codes, which are optimal with respect to this measure.**

***Index Terms*—Multiple frequency shift key modulation, power line communications, narrowband noise, equitable symbol weight codes.**

1. Introduction

POWER line communications (PLC) is a technology that enables the transmission of data over electric power lines. It was started in the 1910's for voice communication [2], and used in the 1950's in the form of ripple control for load and tariff management in power distribution. With the emergence of the Internet in the 1990's, research into broadband PLC gathered pace as a promising technology for Internet access and local area networking, since the electrical grid infrastructure provides "last mile" connectivity to premises and capillarity within premises. Recently, there has been a renewed interest in high-speed narrowband PLC due to applications in sustainable energy strategies, specifically in smart grids (see [3]–[6]).

However, power lines present a difficult communications environment and overcoming permanent narrowband disturbance has remained a challenging problem [7]–[9]. Vinck [7] addressed this problem by showing that multiple frequency shift keying (MFSK) modulation, in conjunction with the use of a permutation code having minimum (Hamming) distance d, is able to correct up to $d-1$ errors due to narrowband noise. Since then, more general codes such as constant-composition codes (see [10]–[19]), frequency permutation arrays (see [18], [20]), and injection codes (see [21]) have been considered as possible replacements for permutation codes in PLC. Versfeld *et al.* [22], [23] later introduced the notion of 'same-symbol weight' (henceforth, termed as *symbol weight*) of a code as a measure of the capability of a code in dealing with narrowband noise. They also showed empirically that low symbol weight cosets of Reed-Solomon codes outperform normal Reed-Solomon codes in the presence of narrowband noise and additive white Gaussian noise. Sizes of symbol-weight spaces were investigated by Chee *et al.* [24] recently.

Unfortunately, symbol weight alone is not sufficient to capture the performance of a code in dealing with permanent narrowband noise. The purpose of the paper is to extend the analysis of Vinck's coded modulation scheme based on permutation codes (see [7], [25, Subsection 5.2.4]) to general codes. In the process, we introduce an additional new parameter that more precisely captures a code's performance against permanent narrowband noise. This parameter is related to *symbol equity*, the uniformity of frequencies of symbols in each codeword. Codes designed taking into account this new parameter, or *equitable symbol weight codes*, are shown to perform better than general ones.

The current proposed standards, such as ITU-T Recommendation G.9902 (G.hnem) and IEEE P1901.2, for communication over narrowband power line channel use Orthogonal Frequency Division Multiplexing (OFDM) based modulation schemes instead of FSK based schemes. In contrast to MFSK scheme which uses only one frequency at a time, OFDM uses multiple frequencies at the same time for transmitting information. Preliminary results on extensions of the current work to use multiple frequencies are presented in [26]. Further investigations and comparisons with current OFDM based schemes are an interesting avenue for future research. Finally, we remark that the notion of symbol equity discussed in this work is also applicable to systems where criss-cross types of errors are encountered [27].

The outline of the rest of the paper is as follows. In Section 2 we introduce the basic definitions and notation. In Section 3 we introduce the noise model and the criterion under which correct decoding can be performed. In particular, we introduce a new parameter that captures how well a code can perform under narrowband noise. In Section 4 we show that equitable symbol weight codes are optimal with respect to this new parameter. We present some simulation results in Section 5 to compare the performance of equitable symbol weight codes with other block codes previously studied in the literature.

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2. Preliminaries

We denote the set of integers and positive integers by $\mathbb{Z}$ and $\mathbb{Z}_{>0}$ respectively. We denote the set $\{1,\ldots,n\}$ by the notation $[n]$. For a finite set X, the collection of all subsets of X, or the *power set* of X, is denoted by 2^X.

Let T be an index set and $\mathcal{X}$ be a set of *symbols*. We denote *a sequence or a vector with index set* T by $(u_t : t \in T, u_t \in \mathcal{X})$. In contrast, we denote a *multiset* by angled brackets, that is, $\langle u_t : t \in T\rangle$. For the latter, when more convenient, the exponential notation $\langle u_1^{t_1} u_2^{t_2} \cdots u_n^{t_n}\rangle$ is used to describe a multiset with exactly t_i elements u_i, $i \in [n]$.

When $|\mathcal{X}| = q$, a *q-ary code* $\mathcal{C}$ of *length* n over the *alphabet* $\mathcal{X}$ is a subset of $\mathcal{X}^n$. Elements of $\mathcal{C}$ are called *codewords*. The *size* of $\mathcal{C}$ is the number of codewords in $\mathcal{C}$. For $i \in [n]$, the ith *coordinate* of a codeword u is denoted by u_i.

A. Symbol weight

Let $\mathsf{u} \in \mathcal{X}^n$. For $\sigma \in \mathcal{X}$, $w_\sigma(\mathsf{u})$ is the number of times the symbol σ appears among the coordinates of u, that is,

$$w_\sigma(\mathsf{u}) = |\{i \in [n] : \mathsf{u}_i = \sigma\}|.$$

The *symbol weight* of u is

$$\mathrm{swt}(\mathsf{u}) = \max_{\sigma\in\mathcal{X}} w_\sigma(\mathsf{u}).$$

A code has *bounded symbol weight* r if the maximum symbol weight of all its codewords is r. A code $\mathcal{C}$ has *constant symbol weight* r if all its codewords have symbol weight exactly r. For any $\mathsf{u} \in \mathcal{X}^n$, observe that $\mathrm{swt}(\mathsf{u}) \geq \lceil n/q\rceil$. A code has *minimum symbol weight* if it has constant symbol weight $\lceil n/q\rceil$.

A codeword $\mathsf{u} \in \mathcal{X}^n$ is said to have *equitable symbol weight* if $w_\sigma(\mathsf{u}) \in \{\lfloor n/q\rfloor, \lceil n/q\rceil\}$ for all $\sigma \in \mathcal{X}$. In other words, if $r = \lceil n/q\rceil$, then every symbol appears r or $r-1$ times in u. If all the codewords of $\mathcal{C}$ have equitable symbol weight, then the code $\mathcal{C}$ is called an *equitable symbol weight code*. Every equitable symbol weight code is hence a minimum symbol weight code.

B. Composition and Partition

The *composition* of $\mathsf{u} \in \mathcal{X}^n$ is the sequence $(w_\sigma(\mathsf{u}) : \sigma \in \mathcal{X})$, and the *partition* of u is the multiset $\langle w_\sigma(\mathsf{u}) : \sigma \in \mathcal{X}\rangle$.

Fix a multiset of nonnegative numbers $\langle c_\sigma : \sigma \in \mathcal{X}\rangle$ such that $\sum_{\sigma\in\mathcal{X}} c_\sigma = n$. A code $\mathcal{C}$ is a *constant composition code with composition* $(c_\sigma : \sigma \in \mathcal{X})$ if all words in $\mathcal{C}$ have composition $(c_\sigma : \sigma \in \mathcal{X})$. Similarly, a code $\mathcal{C}$ is a *constant partition code with partition* $\langle c_\sigma : \sigma \in \mathcal{X}\rangle$ if all words in $\mathcal{C}$ have partition $\langle c_\sigma : \sigma \in \mathcal{X}\rangle$.

Clearly, a constant composition code is necessarily a constant partition code. The following example demonstrates that the converse is not true.

Example 2.1. The code $\{(1,2,3),(2,3,4),(3,4,1),(4,1,2)\}$ is a constant partition code with partition $\langle 1^3 0\rangle$, since in each code word three symbols appear once each, and one symbol does not appear. However, the words have different compositions.

We show that an equitable symbol weight code is necessarily a constant partition code with minimum symbol weight.

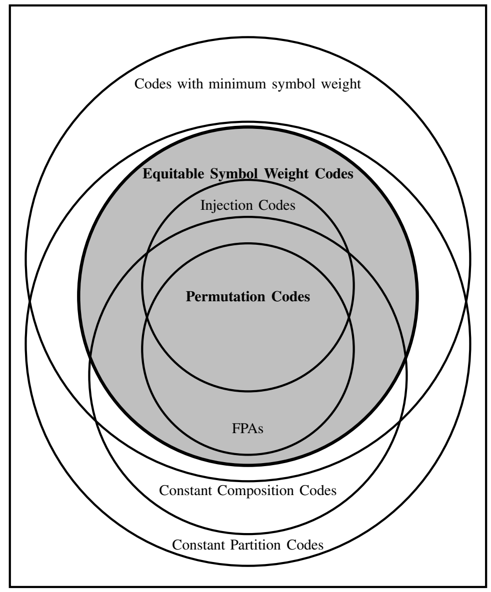

Fig. 1. Generalizations of permutation codes.

This follows from the next lemma that states that for any $\mathsf{u} \in \mathcal{X}^n$ having equitable symbol weight, the number of symbols occurring with frequency $\lceil n/q\rceil$ in u is uniquely determined. Hence, the frequencies of symbols in an equitable symbol weight codeword are as uniformly distributed as possible and the partition of the codeword is fixed.

Lemma 2.1. Let $\mathsf{u} \in \mathcal{X}^n$, $r = \lceil n/q\rceil$, and $t = qr - n$. If u has equitable symbol weight, then u has partition $\langle r^{q-t}(r-1)^t\rangle$

Proof: Let $x = |\{\sigma \in \mathcal{X} : w_\sigma(\mathsf{u}) = r\}|$ and $y = |\{\sigma \in \mathcal{X} : w_\sigma(\mathsf{u}) = r-1\}|$. Then the following equations hold:

$$x + y = q, \text{ and } rx + (r-1)y = n.$$

Solving this set of equations gives the lemma. ■

Using the above notation, we observe that equitable symbol weight codes are generalizations of certain classes of codes which have been studied in PLC applications. For example, if $q|n$, then an equitable symbol weight code has constant partition $\langle (n/q)^q\rangle$, which is known as a frequency permutation array (FPA). If $n \leq q$ then an equitable symbol weight code has constant partition $\langle 1^n 0^{q-n}\rangle$, which is called an injection code. Finally, if $n = q$, then all definitions coincide to give the definition of a permutation code.

C. Hamming Distance

Consider the space $\mathcal{X}^n$ with the distances between words measured in terms of Hamming distance. A q-ary code of length n and distance d is called an $(n,d)_q$*-code*, while a q-ary code of length n having bounded symbol weight r and distance d is called an $(n,d,r)_q$*-symbol weight code*, and a q-ary equitable symbol weight code of length n and distance d is called an $(n,d)_q$*-equitable symbol weight code*.

Remark 1. The notion of symbol equity used here differs from the notion of symbol equity that is used in Swart and Ferriera [28]. In that work, the authors consider the code-matrix of the code (the matrix whose rows consist of all the codewords), and show that an equal distribution of symbols in each column of the code-matrix results in the maximum possible separation between all the codewords. This notion of symbol equity also appears in the computation of the Plotkin bound on codes. In contrast, the symbol equity discussed in this work considers the distribution of symbols in every codeword of the code, i.e., in every row of the code-matrix.

3. CORRECTING NOISE WITH MFSK MODULATION

In coded modulation for power line communications [7], a q-ary code of length n is used, whose symbols are modulated using q-ary MFSK. The receiver demodulates the received signal using an envelope detector to obtain an output, which is then decoded by a decoder.

Four detector/decoder combinations are possible: classical, modified classical, hard-decision threshold, and soft-decision threshold (see [25] for details). A soft-decision threshold detector/decoder requires exact channel state knowledge and is therefore not useful if we do not have channel state knowledge. Henceforth, we consider the hard-decision threshold detector/decoder here, since it contains more information about the received signal compared to the classical and modified classical ones. We remark that in the case of the hard-decision threshold detector/decoder, the decoder used is a *minimum distance decoder*.

Let $\mathcal{C}$ be an $(n,d)_q$-code over alphabet $\mathcal{X}$, and let $\mathsf{u} = (\mathsf{u}_1,\dots,\mathsf{u}_n)$ be a codeword transmitted over the PLC channel where the symbol u_i is transmitted at discrete time instance i for $i \in [n]$. The received signal (which may contain errors caused by noise) is demodulated to give an output $\mathsf{v} = (\mathsf{v}_1,\mathsf{v}_2,\dots,\mathsf{v}_n)$ in which each v_i is a subset of $\mathcal{X}$. The errors that arise from the different types of noise in the channel (see [25, pp. 222–223]) have the following effects on the output of the detector.

(i) Narrowband noise at a particular frequency introduces a symbol at several consecutive discrete time instances of the transmitted signals. The narrowband noise affects only a part of the transmission that occurs at discrete time instances from $i=1$ to $i=n$. Hence, narrowband noise of duration l affects up to l consecutive positions in the discrete time instances from $i=1$ to $i=n$, depending on whether the noise started prior to or during the current transmission. Narrowband noise may be present simultaneously at multiple frequencies corresponding to different symbols.
Let $1 \le e \le q$ and $l \in \mathbb{Z}_{>0}$. If e *narrowband noise errors of duration* l occur, then there is a set $\mathcal{Y} \subseteq \mathcal{X}$ consisting of e symbols, and e corresponding starting instances $\{i_\sigma \le n : \sigma \in \mathcal{Y}\}$ such that for $\sigma \in \mathcal{Y}$,
$$\sigma \in \mathsf{v}_i \text{ for } \max\{1, i_\sigma\} \le i \le \min\{i_\sigma + l - 1, n\}.$$
(ii) A signal fading error results in the absence of a symbol in the received signal. Let $1 \le e \le q$. If e *signal fading errors* occur, then there are e symbols, none of which appears in any v_i, that is, $(\cup_{i=1}^n \mathsf{v}_i) \cap \mathcal{Y} = \varnothing$ for some $\mathcal{Y} \subseteq \mathcal{X},\ |\mathcal{Y}| = e$.
(iii) Impulse noise results in the entire set of symbols being received at a certain discrete time instance. Let $1 \le e \le n$. If e *impulse noise errors* occur, then there is a set $\Pi \subseteq [n]$ consisting of e positions such that $\mathsf{v}_i = \mathcal{X}$ for all $i \in \Pi$.
(iv) An insertion error results in an unwanted symbol in the received signal. Let $1 \le e \le n(q-1)$. If e *insertion errors* occur, then there is a set $\Omega \subseteq [n] \times \mathcal{X}$ of size e such that for each $(i,\sigma) \in \Omega$, v_i contains σ and $\sigma \ne \mathsf{u}_i$.
(v) A deletion error results in the absence of a transmitted symbol in the received signal. Let $1 \le e \le n$. If e *deletion errors* occur, then there is a set $\Pi \subseteq [n]$ consisting of e positions such that v_i does not contain u_i for all $i \in \Pi$.

Both insertion and deletion errors are due to *background noise*. This definition of insertion and deletion error is different from the errors that arise in an "insertion-deletion channel" [29].

Example 3.1. Suppose $\mathsf{u} = (1,2,3,4)$.
(i) Narrowband noise can start prior to or during the transmission of u. Narrowband noise error of duration 4 at symbol 1 starting at discrete time instance $i=-1$ results in detector output $\mathsf{v} = (\{1\},\{1,2\},\{3\},\{4\})$, while the same narrowband noise error starting at discrete time instance $i=3$ results in detector output $\mathsf{v} = (\{1\},\{2\},\{1,3\},\{1,4\})$.
(ii) The same detector output can arise from different combinations of error types. A signal fading error of symbol 1 and a deletion error at position 1 would each result in the same detector output of $\mathsf{v} = (\varnothing,\{2\},\{3\},\{4\})$.

Recall that $2^{\mathcal{X}}$ denotes the power set of $\mathcal{X}$. For a codeword $\mathsf{u} \in \mathcal{X}^n$ and an output $\mathsf{v} \in \left(2^{\mathcal{X}}\right)^n$, define
$$d(\mathsf{u},\mathsf{v}) = |\{i : \mathsf{u}_i \notin \mathsf{v}_i\}|.$$
Note that in this context, we identify $\mathsf{c} \in \mathcal{X}^n$ with $(\{\mathsf{c}_1\},\{\mathsf{c}_2\},\dots,\{\mathsf{c}_n\}) \in \left(2^{\mathcal{X}}\right)^n$, so that $d(\mathsf{u},\mathsf{c})$ gives the Hamming distance between u and c. We also extend the definition of distance so that for $\mathcal{C} \subseteq \mathcal{X}^n$, we have $d(\mathcal{C},\mathsf{v}) = \min_{\mathsf{u}\in\mathcal{C}} d(\mathsf{u},\mathsf{v})$. Given $\mathsf{v} \in (2^{\mathcal{X}})^n$, a minimum distance decoder (for a code $\mathcal{C}$) outputs a codeword $\mathsf{u} \in \mathcal{C}$ which has the smallest distance to v, that is, a minimum distance decoder returns an element of
$$\underset{\mathsf{u}\in\mathcal{C}}{\arg\min}\ d(\mathsf{u},\mathsf{v}) := \{\mathsf{u} \in \mathcal{C} : d(\mathsf{u},\mathsf{v}) \le d(\mathsf{u}',\mathsf{v})\ \forall \mathsf{u}' \in \mathcal{C}\}. \quad (1)$$
In the following, we study the conditions under which a minimum distance decoder outputs the correct codeword, that is, when $\underset{\mathsf{u}'\in\mathcal{C}}{\arg\min}\ d(\mathsf{u}',\mathsf{v}) = \{\mathsf{u}\}$. This is equivalent to saying that the decoder correctly outputs u if and only if $d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{v}) > d(\mathsf{u},\mathsf{v})$.

Let $d' = d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{u})$. Since $\mathcal{C}$ has distance d, we have $d' \ge d$. Observe the following:

- Let $1 \le e \le n$. If e impulse noise errors occur, then in e coordinates all the symbols occur. Therefore, those e coordinates do not contribute to the distance between v and any codeword. Hence, we get
$$d(\mathsf{u},\mathsf{v}) = 0 \quad\text{and}\quad d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{v}) \ge d' - e.$$
- Let $1 \le e \le n(q-1)$. If e insertion errors occur, then there are at most e coordinates which do not contribute to

the distance between v and some codeword in the code. Hence, we get

$$d(\mathsf{u},\mathsf{v}) = 0 \quad \text{and} \quad d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{v}) \geq d' - e.$$

- Let $1 \leq e \leq n$. If e deletion errors occur, then there are exactly e coordinates where the transmitted codeword u differs from v. Any other codeword still differs from v in at least d' coordinates. Therefore, we get

$$d(\mathsf{u},\mathsf{v}) = e \quad \text{and} \quad d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{v}) \geq d'.$$

For errors due to narrowband noise we introduce a quantity that measures how many coordinates of any codeword in the code are affected by the noise. Specifically, a narrowband noise at the frequency corresponding to symbol σ can affect up to n coordinates in a codeword, depending on the number of times the symbol σ appears in the codeword. If narrowband noise is present in the set of symbols $\mathcal{Y} \subseteq \mathcal{X}$, then the maximum number of entries of any codeword c that can be affected by the noise is $\sum_{\sigma\in\mathcal{Y}} w_\sigma(\mathsf{c})$. Therefore, we define

$$E(e;\mathcal{C}) \triangleq \max_{\mathsf{c}\in\mathcal{C},\, \mathcal{Y}\subseteq\mathcal{X},\, |\mathcal{Y}|=e} \sum_{\sigma\in\mathcal{Y}} w_\sigma(\mathsf{c}). \tag{2}$$

The expression $E(e;\mathcal{C})$ measures the maximum number of coordinates, over all codewords in $\mathcal{C}$ that are affected by e narrowband noise. Equation (2) assumes that the duration of the narrowband noise is at least n and that it is present in all the coordinates of the codeword transmitted. In general, narrowband noise of duration l at symbol σ may not be present for the full duration of the codeword. In Appendix A we show that it suffices to consider narrowband noise of duration n since it measures the maximum effect of narrowband noise on the codewords.

Recall that $d' = d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{u})$. From the definition of $E(e;\mathcal{C})$, it is clear that the distance between any codeword, other than the transmitted codeword u, and the output v decreases by $E(e;\mathcal{C})$. Similarly, in the presence of a fading error the distance between u and v increases by at most $E(e;\mathcal{C})$. Therefore we get the two conditions mentioned below.

- Let $1 \leq e \leq q$. If e narrowband noise errors occur, then

$$d(\mathsf{u},\mathsf{v}) = 0 \quad \text{and} \quad d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{v}) \geq d' - E(e;\mathcal{C}).$$

- Let $1 \leq e \leq q$. If e signal fading errors occur, then

$$d(\mathsf{u},\mathsf{v}) \leq E(e;\mathcal{C}) \quad \text{and} \quad d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{v}) \geq d'.$$

Hence, if we denote by e_{N}, e_{F}, e_{IMP}, e_{INS}, and e_{DEL} the number of errors due to narrowband noise, signal fading, impulse noise, insertion, and deletion, respectively, we have

$$d(\mathsf{u},\mathsf{v}) \leq e_{\mathsf{DEL}} + E(e_{\mathsf{F}};\mathcal{C}),$$
$$d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{v}) \geq d' - e_{\mathsf{IMP}} - e_{\mathsf{INS}} - E(e_{\mathsf{N}};\mathcal{C}).$$

Now,

$$\begin{aligned} &d(\mathsf{u},\mathsf{v}) - d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{v}) \\ &\quad\leq (e_{\mathsf{DEL}} + E(e_{\mathsf{F}};\mathcal{C})) - (d' - e_{\mathsf{IMP}} - e_{\mathsf{INS}} - E(e_{\mathsf{N}};\mathcal{C})) \\ &\quad= e_{\mathsf{DEL}} + e_{\mathsf{IMP}} + e_{\mathsf{INS}} + E(e_{\mathsf{F}};\mathcal{C}) + E(e_{\mathsf{N}};\mathcal{C}) - d'. \end{aligned} \tag{3}$$

Under the condition

$$e_{\mathsf{DEL}} + e_{\mathsf{IMP}} + e_{\mathsf{INS}} + E(e_{\mathsf{F}};\mathcal{C}) + E(e_{\mathsf{N}};\mathcal{C}) < d,$$

the inequality (3) reduces to $d(\mathsf{u},\mathsf{v}) < d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{v})$, which implies correct decoding.

On the other hand, if

$$e_{\mathsf{DEL}} + e_{\mathsf{IMP}} + e_{\mathsf{INS}} + E(e_{\mathsf{F}};\mathcal{C}) + E(e_{\mathsf{N}};\mathcal{C}) \geq d,$$

say $e_{\mathsf{IMP}} = d$, and $\mathsf{u},\mathsf{w} \in \mathcal{C}$ is such that $d(\mathsf{u},\mathsf{w}) = d$ (since $\mathcal{C}$ has distance d, u,w must exist), then $d' = d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{u}) = d$, and we have $d(\mathsf{u},\mathsf{v}) - d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{v}) \leq d - d' = 0$. In this case, the correctness of the decoder output cannot be guaranteed. We therefore have the following theorem.

Theorem 3.1. Let $\mathcal{C}$ be an $(n,d)_q$-code over alphabet $\mathcal{X}$. Let $e_{\mathsf{DEL}}, e_{\mathsf{IMP}}, e_{\mathsf{INS}} \in [n]$, $e_{\mathsf{N}}, e_{\mathsf{F}} \in [q]$. Then $\mathcal{C}$ is able to correct e_{N} narrowband noise errors, e_{F} signal fading errors, e_{IMP} impulse noise errors, e_{INS} insertion errors, and e_{DEL} deletion errors if and only if

$$e_{\mathsf{DEL}} + e_{\mathsf{IMP}} + e_{\mathsf{INS}} + E(e_{\mathsf{F}};\mathcal{C}) + E(e_{\mathsf{N}};\mathcal{C}) < d.$$

Therefore, the parameters n, q, d, and r (symbol weight) of a code are insufficient to characterize the total error-correcting capability of a code in a PLC system using MFSK, since $E(e;\mathcal{C})$ cannot be specified by n, q, d, and r alone. We now introduce an additional new parameter that together with n, q, and d, more precisely captures the error-correcting capability of a code for PLC using MFSK.

Definition 3.1. Let $\mathcal{C}$ be a code of distance d. The *narrowband noise error-correcting capability* of $\mathcal{C}$ is

$$c(\mathcal{C}) = \min\{e : E(e;\mathcal{C}) \geq d\}.$$

From Theorem 3.1 we infer that a code $\mathcal{C}$ can correct up to $c(\mathcal{C})-1$ narrowband noise errors. In general, the minimum value of $c(\mathcal{C})$ is about d/r if all the symbols occur exactly r times, and the maximum value of $c(\mathcal{C})$ is at most d if all the symbols appear once. Therefore, for a code $\mathcal{C}$ with bounded symbol weight r, we have $\lceil d/r \rceil \leq c(\mathcal{C}) \leq \min\{d,q\}$. However, the gap between the upper and lower bounds can be large. Furthermore, the lower bound can be attained, giving codes of low resilience against narrowband noise, as is shown in the following example.

Example 3.2. The code

$$\mathcal{C} = \{(\underbrace{1,\ldots,1}_{r\text{ times}},2,3,4,\ldots,q),(\underbrace{2,\ldots,2}_{r\text{ times}},1,3,4,\ldots,q)\}$$

is a $(q+r-1,r+1,r)_q$-symbol weight code with narrowband noise error-correcting capability $c(\mathcal{C}) = \lceil d/r \rceil = 2$.

In the rest of the paper, we write $E(\mathcal{C})$ when we want to consider $E(e;\mathcal{C})$ as a function of e, $E(\mathcal{C}) : [q] \to [n]$, for a specific code $\mathcal{C}$. In the next section, we provide a tight upper bound for $c(\mathcal{C})$ and demonstrate that equitable symbol weight codes attain this upper bound.

4. $E(\mathcal{C})$ and Equitable Symbol Weight Codes

In general, for a PLC system, narrowband noise may occur with different durations. However, because of the result in Lemma A.1, in the rest of this section we consider only narrowband noise of duration n for analysis. In the rest of the section, we then demonstrate the optimality of equitable symbol weight codes with respect to parameter $E(\mathcal{C})$.

A. Relation with Symbol Weight and Partition

Symbol weight provides an estimate for $E(\mathcal{C})$. Specifically, if $\mathcal{C}$ is a code of length n with bounded symbol weight r, then $E(1;\mathcal{C}) = r$, and for $e > 1$ the minimum value possible is $r+e-1$ if any other symbol occurs exactly once. Therefore, $E(e;\mathcal{C}) \geq \min\{n, r+e-1\}$.

On the other hand, if $\mathcal{C}$ is a constant partition code with partition $\langle c_\sigma : \sigma \in \mathcal{X}\rangle$, $E(\mathcal{C})$ can be determined precisely. Assume $\mathcal{X} = [q]$ and $c_1 \geq c_2 \geq \cdots \geq c_q$, then $E(e;\mathcal{C})$ is the sum of e largest symbol weights in any codeword, i.e.,

$$E(e;\mathcal{C}) = \sum_{i=1}^{e} c_i \text{ for all } e \in [q].$$

Further, suppose that $\mathcal{C}$ is an equitable symbol weight code. Then from Lemma 2.1, $\mathcal{C}$ has constant partition $\langle r^{q-t}(r-1)^t\rangle$, where $r = \lceil n/q \rceil$ and $t = qr - n$. Hence,

$$E(e;\mathcal{C}) = \begin{cases} re, & \text{if } e \leq q-t, \\ r(q-t) + (e-q+t)(r-1), & \text{if } q-t < e \leq q. \end{cases}$$

B. Importance of Symbol Equity

For $c(\mathcal{C})$ to be large, $E(\mathcal{C})$ must grow slowly as a function of e. We seek codes $\mathcal{C}$ for which $E(\mathcal{C})$ grows as slowly as possible. In this subsection we show that the minimum growth of $E(\mathcal{C})$ is achieved when the maximum symbol weight in any codeword of the code is at most $\lceil n/q \rceil$, i.e., the symbols are equitably distributed in any codeword. Fix n, q, and let $\mathcal{F}_{n,q}$ be the (finite) family of functions

$$\mathcal{F}_{n,q} = \{E(\mathcal{C}) : \mathcal{C} \text{ is a } q\text{-ary code of length } n\}.$$

If $f \in \mathcal{F}_{n,q}$, then f is a monotone increasing function with $f(q) = n$. We say that $f \prec g$ if

$$\text{there exists } e' \in [q] \text{ with } f(e) = g(e) \text{ for } e \leq e'-1, \text{ and } f(e') < g(e'). \quad (4)$$

Define the total order $\preceq$ on $\mathcal{F}_{n,q}$ so that $f \preceq g$ if either $f(e) = g(e)$ for all $e \in [q]$ or $f \prec g$.

The following proposition states that the total order $\preceq$, in some sense, orders codes of same length and alphabet size in accordance to their capabilities in a PLC system.

Proposition 4.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be $(n,d)_q$-codes. Suppose $E(\mathcal{C}) \prec E(\mathcal{C}')$ with e' satisfying equation (4). If $E(e';\mathcal{C}) < d$, then there exists a set of errors that $\mathcal{C}$ is able to correct but $\mathcal{C}'$ is unable to correct.

Proof: Consider e' narrowband noise errors of duration n and $d - E(e';\mathcal{C}) - 1$ impulse errors. Then $E(e';\mathcal{C}) + (d - E(e';\mathcal{C}) - 1) < d$, but $E(e';\mathcal{C}') + (d - E(e';\mathcal{C}) - 1) \geq d$. The proposition then follows from Theorem 3.1. ■

Hence we seek the least element in $\mathcal{F}_{n,q}$ with respect to the total order $\preceq$.

Proposition 4.2. Let $f^*_{n,q} : [q] \to [n]$ be defined by

$$f^*_{n,q}(e) = \begin{cases} re, & \text{if } 1 \leq e \leq q-t, \\ r(q-t) + (e-q+t)(r-1), & \text{otherwise,} \end{cases}$$

where $r = \lceil n/q \rceil$ and $t = qr - n$. Then $f^*_{n,q}$ is the unique least element in $\mathcal{F}_{n,q}$ with respect to the total order $\preceq$.

Proof: Since $\preceq$ is total, it suffices to establish that $f^*_{n,q} \preceq f$ for all $f \in \mathcal{F}_{n,q}$, and that $f^*_{n,q} \in \mathcal{F}_{n,q}$.

Let $f = E(\mathcal{C}) \in \mathcal{F}_{n,q}$, where $\mathcal{C}$ is a q-ary code of length n over the alphabet $[q]$. Let $\mathsf{u} \in \mathcal{C}$. By permuting symbols if necessary, we may assume that $w_1(\mathsf{u}) \geq w_2(\mathsf{u}) \geq \cdots \geq w_q(\mathsf{u})$. We show that for all $e \in [q]$,

$$\sum_{i=1}^{e} w_i(\mathsf{u}) \geq f^*_{n,q}(e). \quad (5)$$

Suppose on the contrary that $\sum_{i=1}^{e} w_i(\mathsf{u}) < f^*_{n,q}(e)$ for some $e \in [q]$. If $e \leq q-t$, then we have $\sum_{i=1}^{e} w_i(\mathsf{u}) < re$ and $r-1 \geq w_e(\mathsf{u}) \geq w_j(\mathsf{u})$ for $j \geq e+1$. Hence,

$$n = \sum_{i=1}^{q} w_i(\mathsf{u}) < re + (q-e)(r-1) = qr - q + e \leq qr - t = n,$$

a contradiction.

Similarly, when $e > q-t$, we have $\sum_{i=1}^{e} w_i(\mathsf{u}) < r(q-t) + (e-q+t)(r-1)$ and $r-1 \geq w_e(\mathsf{u}) \geq w_j(\mathsf{u})$ for $j \geq e+1$. Hence,

$$\begin{aligned} n &= \sum_{i=1}^{q} w_i(\mathsf{u}) \\ &< r(q-t) + (e-q+t)(r-1) + (q-e)(r-1) \\ &= qr - t = n, \end{aligned}$$

also a contradiction. Hence, (5) holds. This then implies $E(e;\mathcal{C}) \geq f^*_{n,q}(e)$ for all $e \in [q]$, and consequently $f \succeq f^*_{n,q}$.

The proposition then follows by noting that $f^*_{n,q} \in \mathcal{F}_{n,q}$, since $E(\mathcal{C}) = f^*_{n,q}$ when $\mathcal{C}$ is a q-ary equitable symbol weight code of length n. ■

Corollary 4.1. $\mathcal{C}$ is a q-ary equitable symbol weight code of length n if and only if $E(\mathcal{C}) = f^*_{n,q}$.

Proof: If $\mathcal{C}$ is a q-ary equitable symbol weight code of length n, we have already determined that $E(\mathcal{C}) = f^*_{n,q}$. Hence, it only remains to show that $E(\mathcal{C}) = f^*_{n,q}$ implies $\mathcal{C}$ is a q-ary equitable symbol weight code of length n. Let $\mathsf{u} \in \mathcal{C}$ and we follow the notation in the proof of Proposition 4.2. Equality holds in (5) if and only if $w_i(\mathsf{u}) = r$ for $1 \leq i \leq q-t$ and $w_i(\mathsf{u}) = r-1$, otherwise. That is, u has equitable symbol weight. Hence, $\mathcal{C}$ is an equitable symbol weight code. ■

It follows that an equitable symbol weight code $\mathcal{C}$ gives $E(\mathcal{C})$ of the slowest growth rate. From Proposition 4.1, this is the desired condition for correcting as many narrowband noise and signal fading errors as possible.

We end this section with a tight upper bound on $c(\mathcal{C})$.

Corollary 4.2. Let $\mathcal{C}$ be an $(n,d)_q$-code. Then

$$c(\mathcal{C}) \leq \min\{e : f^*_{n,q}(e) \geq d\},$$

and equality is achieved when $\mathcal{C}$ is an equitable symbol weight code.

Proof: Let $c' = \min\{e : f^*_{n,q}(e) \geq d\}$. Observe that

$$E(c';\mathcal{C}) \geq f^*_{n,q}(c') \geq d.$$

Hence, by minimality of $c(\mathcal{C})$, we have $c(\mathcal{C}) \leq c'$. The second part of the statement follows from Corollary 4.1. ■

The results in this section establish that an equitable symbol weight code has the best narrowband noise and signal fading error-correcting capability, among codes of the same distance and symbol weight.

5. Simulation Results

In this section, we study the performance of equitable symbol weight codes in a simulated setup. The setup is as follows. We transmit with a code of length n over alphabet $\mathcal{X}$. Let p be a real number between 0 and 1 and $L = \{bn : b \in [10]\}$. We simulate a PLC channel with the following characteristics:

(i) for each $\sigma \in \mathcal{X}$, narrowband noise error[1] of duration $l \in L$ occurs at symbol σ with probability p,
(ii) for each $\sigma \in \mathcal{X}$, a signal fading error occurs at symbol σ with probability Q,
(iii) for each $i \in [n]$, an impulse noise error occurs at coordinate i with probability Q, and
(iv) for each $(\sigma, i) \in \mathcal{X} \times [n]$, an insertion/deletion error occurs at symbol σ and coordinate i with probability Q.

These errors occur independently.

We choose 10^5 random codewords (with repetition) from each code to transmit through the simulated PLC channel. At the receiver, we decode the detector output v to the codeword u' using the minimum distance decoder defined in equation (1). The number of symbols in error is then $d(\mathsf{u}', \mathsf{u})$ and the *symbol error rate* is the ratio of the total number of symbols in error to the total number of symbols transmitted.

Decoding with narrowband noise detection: Versfeld *et al.* [22], [23] introduced a method to detect narrowband noise in order to enhance the error correction capability of the detector introduced in Section 3, when used with bounded distance decoding. Based on the energy metrics obtained at each time slot for each frequency, they first determine the presence of narrowband interference and if so, the metrics of the corresponding frequency are set to zero. Depending on the detector/decoder combination, a signal is sent to the decoder. Specifically, consider narrowband noise detection with the use of an (n, d, r)-symbol weight code. If the number of discrete time instances in which a particular symbol appears, exceeds $\lfloor (n + r)/2 \rfloor$, the particular symbol is removed from the coordinates in which it occurs. We describe an algorithm to detect and remove narrowband noise in Algorithm 1.

A. Minimum Symbol Weight Codes

We exhibit the difference in performance between equitable symbol weight codes and (non-equitable) minimum symbol weight codes. Specifically, we consider the codes of various lengths and relative distances in Table I.

In Fig. 2 we show the difference between the performance of the codes for varying probability of narrowband noise. The different plots correspond to the probability of background noise, impulse noise and fading fixed at $Q \in \{0.1, 0.075, 0.05, 0.025, 0.01\}$. The solid lines correspond to equitable symbol weight codes and the dotted lines correspond to minimum symbol weight codes. Only for this particular simulation 10^7 codewords are transmitted. In the simulations we detect the presence of narrowband noise[2] using Algorithm 1.

[1]The choice of L is similar to that of the narrowband noise model in the setup of Verfeld *et al.* [22], [23].

Input: Detector Output, $\mathsf{v} \in (2^{\mathcal{X}})^n$
Output: Modified $\mathsf{v} \in (2^{\mathcal{X}})^n$

$\tau \leftarrow \lfloor (n + r)/2 \rfloor$;
for $\sigma \in \mathcal{X}$ **do**
  if $|\{i : \sigma \in \mathsf{v}_i\}| > \tau$ **then**
    for $i \in [n]$ **do**
      $\mathsf{v}_i \leftarrow \mathsf{v}_i \setminus \{\sigma\}$
    end
  end
end

Algorithm 1: Narrowband noise detection with an (n, d, r)-symbol weight code

For the rest of the simulations we fix $Q = 0.05$. For equitable and minimum symbol weight codes of size 1000, the results of the simulation are displayed in Fig. 3. The solid lines correspond to simulations in which we detect narrowband noise and are labelled by "(NB)". The dashed lines denote simulations without narrowband noise detection. From the results, observe that $\mathrm{ESW}(25, 24, 2)_{17}$ and $\mathrm{ESW}(11, 6, 2)_{10}$ achieve lower symbol error rates compared to $\mathrm{MSW}(25, 24, 2)_{17}$ and $\mathrm{MSW}(11, 6, 2)_{10}$, respectively.

B. Cosets and Subcodes of Reed-Solomon Codes

Versfeld *et al.* [22], [23] showed empirically that using narrowband detection, low symbol weight cosets of Reed-Solomon codes outperform normal Reed-Solomon codes in the presence of narrowband noise and additive white Gaussian noise. We continue this investigation and observe the difference in performance between equitable symbol weight codes and low symbol weight cosets of Reed-Solomon codes. In addition, we consider subcodes of Reed-Solomon codes with low symbol weight. In all these simulations we fix $Q = 0.05$, and vary the probability p of narrowband noise.

Specifically, we consider the codes in Table II. See [22], [23] for the construction of Reed-Solomon coset codes, denoted by RSC. The codes denoted by RSS are subcodes of Reed-Solomon codes. They are obtained by expurgation of a Reed-Solomon code and retaining only the codewords with low symbol weight.

We note that it is not possible for equitable symbol weight codes and Reed-Solomon coset codes of the same minimum distance and length over the same alphabet to be of the same size. Therefore, for each Reed-Solomon coset codes, we make comparisons with an equitable symbol weight code of a larger size, albeit with a smaller distance. However, these equitable symbol weight codes have larger narrowband noise error-correcting capabilities. In addition, we make comparisons with subcodes of Reed-Solomon codes with parameters as close as

[2]As discussed in Section 3, after narrowband noise detection, the multivalued output is given directly to a minimum distance decoder. This deviation from the setup by Versfeld *et al.* (where envelope detection and Viterbi threshold ratio test is applied prior to decoding) means that the results are independent of the choice of demodulation rule.

TABLE I
COMPARISON OF EQUITABLE SYMBOL WEIGHT CODES AND MINIMUM SYMBOL WEIGHT CODES

Code	Length	Distance	Narrowband noise error-correcting capability	Symbol weight	Alphabet size	Size	Remarks
$\mathrm{ESW}(25,24,2)_{17}$	25	24	16	2	17	51	equitable symbol weight
$\mathrm{MSW}(25,24,2)_{17}$	25	24	12	2	17	51	minimum symbol weight
$\mathrm{ESW}(11,6,2)_{10}$	11	6	5	2	10	1000	equitable symbol weight
$\mathrm{MSW}(11,6,2)_{10}$	11	6	3	2	10	1000	minimum symbol weight

TABLE II
COMPARISON OF EQUITABLE SYMBOL WEIGHT CODES AND LOW SYMBOL WEIGHT COSETS AND SUBCODES OF REED-SOLOMON CODES

Code	Length	Distance	Narrowband noise error-correcting capability	Symbol weight	Alphabet size	Size	Remarks
$\mathrm{ESW}(7,5,1)_8$	7	5	5	1	8	336	equitable symbol weight
$\mathrm{RSC}(7,6,2)_8$	7	6	3	2	8	64	coset of Reed-Solomon code
$\mathrm{RSS}(7,5,2)_8$	7	5	3	2	8	336	subcode of Reed-Solomon code
$\mathrm{ESW}(7,2,1)_8$	7	2	2	1	8	20160	equitable symbol weight
$\mathrm{RSC}(7,4,4)_8$	7	4	1	4	8	4096	coset of Reed-Solomon code
$\mathrm{RSS}(7,3,2)_8$	7	3	2	2	8	20160	subcode of Reed-Solomon code
$\mathrm{ESW}(15,11,1)_{16}$	15	11	11	1	16	21120	equitable symbol weight
$\mathrm{RSC}(15,13,3)_{16}$	15	13	5	3	16	4096	coset of Reed-Solomon code
$\mathrm{RSS}(15,12,3)_{16}$	15	12	4	3	16	21120	subcode of Reed-Solomon code

possible to the corresponding equitable symbol weight codes. In particular, we ensure that the subcodes and the equitable symbol weight codes have the same size.

The results of the simulation are displayed in Fig. 4, where we adopt similar conventions as in Fig. 3, and we make the following observations.

(i) While narrowband noise detection in general improves the performance of codes in PLC, it has negligible effect on the performance of equitable symbol weight codes. A natural question is if there is another parameter that measures this improvement and if this parameter is related to symbol equity.

(ii) Equitable symbol weight codes show larger improvement over Reed-Solomon coset codes at higher narrowband noise probabilities. This reflects the relevance of narrowband noise error-correcting capabilities as a measure of performance when the effects of narrowband interference are significant. In contrast, when the effects of narrowband interference are negligible, the classical Hamming distance parameter provides a better measure of performance.

C. Simulation in the presence of cyclostationary noise

By definition, the parameter $c(\mathcal{C})$ of a code $\mathcal{C}$ captures the performance of the code in the presence of narrowband noise and fading. It captures a "worst-case error" performance, similar to how the minimum distance of a code determines the worst-case error performance under bounded distance decoding. A natural question arises about how a code $\mathcal{C}$ with a narrowband noise error correcting capability $c(\mathcal{C})$ performs in the presence of cyclostationary noise (periodically varying noise) compared to a code $\mathcal{C}'$ with a lower value of $c(\mathcal{C}')$. We compare the performance of the equitable symbol weight code $\mathrm{ESW}(25,24,2)_{17}$ and the minimum symbol weight code $\mathrm{MSW}(25,24,2)_{17}$ under the presence of cyclostationary noise.

The setup is as follows. A model for cyclostationary noise in the power line channel is presented in [30]. Gaussian noise

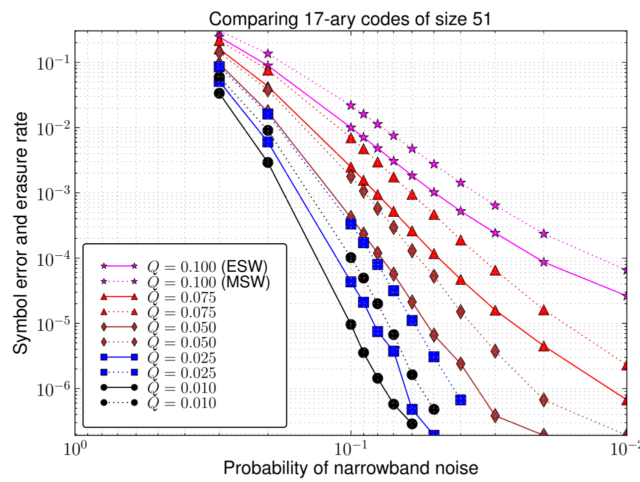

Fig. 2. Comparison of equitable symbol weight codes (solid lines) and minimum symbol weight codes (dashed lines) with varying probabilities of noise.

is generated with instantaneous variance

$$\sigma^2(t) = 0.23 + 1.38\left|\sin\left(2\pi\frac{t}{T_{AC}} - 0.10\right)\right|^{1.91} + 7.17\left|\sin\left(2\pi\frac{t}{T_{AC}} - 0.61\right)\right|^{157000},$$

where $T_{AC} = 1/60\,\mathrm{s}$ is the period of the mains voltage. The instantaneous variance has a period of $T_{AC}/2$, and has an average variance of one, when averaged over this period. The generated Gaussian noise is then passed through a filter with amplitude response $H(f) = \sqrt{a/2}e^{-a|f|/2}$., where $a = 1.2 \times 10^{-5}$. Let the alphabet of the codes be the set $[17]$. We require 17 individual center frequencies to represent each symbol from the alphabet. The transmitted signals are modulated according to the sinusoidal waves

$$s_m(t) = \sqrt{\frac{2E_s}{T_s}}\cos(2\pi f_m t),\ t \in [0, T_s), m \in [17],$$

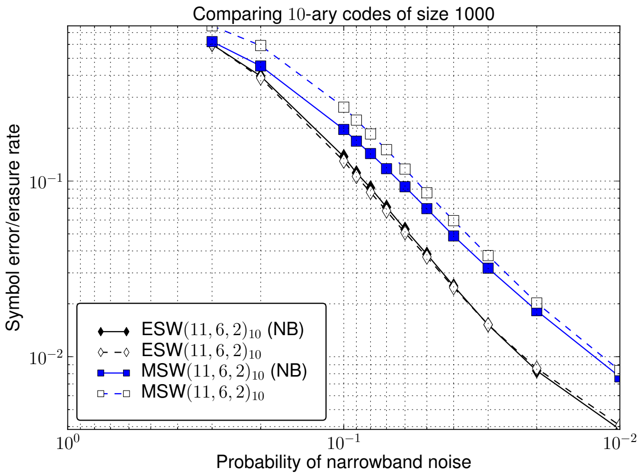

Fig. 3. Comparison of equitable and minimum symbol weight codes.

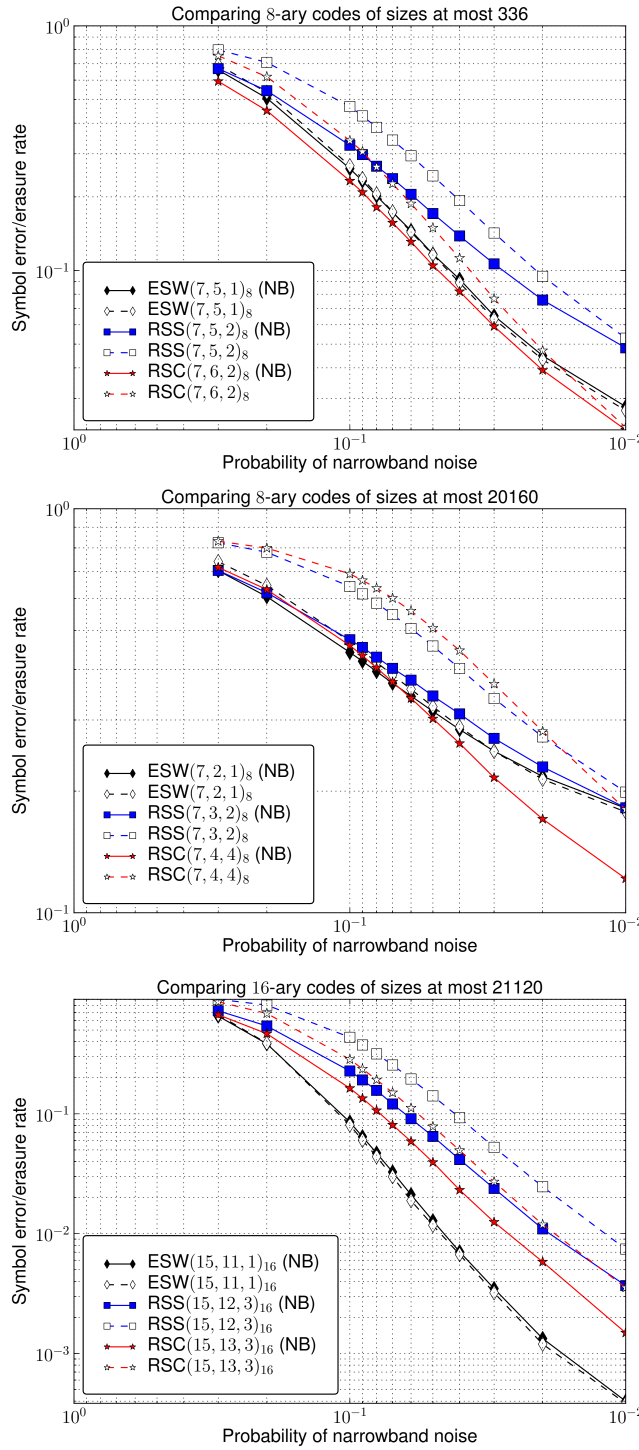

Fig. 4. Comparison of equitable symbol weight codes and low symbol weight cosets and subcodes of Reed-Solomon codes.

where E_s is the symbol energy, T_s is the symbol time period, and f_m are the center frequencies. The time period of each symbol is taken to be $T_s = 1/9 \times T_{AC}/2$, and the signal is sampled at the rate $500 \times 1080\,\text{Hz}$, which is slightly above 500 kHz. This sampling rate is chosen to give the same integral number of samples in each $T_{AC}/2$ period. The center frequency corresponding to the symbol m is taken to be $10.8m\,\text{kHz}$, for $m \in [17]$. This maintains a frequency separation of $10.8\,\text{kHz}$ that is an integral multiple of $1/T_s$ and ensures a correlation of zero between the different signal waveforms. At each center frequency we use a square-law detector and declare one if it detects an energy greater than $E_s/4$, otherwise it declares a zero[3]. While decoding, the codes detect the presence of narrowband noise using Algorithm 1. The output of the simulation is presented in Fig. 5. The horizontal axis corresponds to the signal to average noise ratio (in dB), where the average noise power spectral density is denoted by N_0. We observe that the equitable symbol weight codes outperform minimum symbol weight codes when the noise process is cyclostationary.

6. Conclusion

We have introduced a new code parameter that captures the error-correcting capability of a code with respect to narrowband noise and signal fading. Equitable symbol weight codes are shown to be optimal with respect to this parameter when code length, alphabet size and distance are fixed. We also provide simulations that show equitable symbol weight codes achieve lower symbol error rates as compared to their non-equitable counterparts. These results motivate the study of equitable symbol weight codes as a viable option to handle narrowband noise and signal fading in a PLC channel.

Appendix A
Narrowband Noise of Different Durations and $E(e;\mathcal{C})$

In this Appendix we show that it suffices to consider narrowband noise of length n instead of smaller lengths since it measures the maximum effect of narrowband noise on the codewords. For integers $k, n,\ k \le n$, we use the notation $[k, n] \triangleq \{k, \dots, n\}$. Therefore, given n and for an integer $i_\sigma \le n$, we can write $\{i : \max\{1, i_\sigma\} \le i \le \min\{i_\sigma + l - 1, n\}\} = [i_\sigma, i_\sigma + l - 1] \cap [n]$. For errors due to narrowband noise, we define the following quantity for $\mathcal{Y} \subset \mathcal{X}$, $l \in \mathbb{Z}_{>0}$, $\mathsf{c} \in \mathcal{C}$,

$$E(\mathcal{Y}; l, \mathsf{c}) = \max_{i_\sigma \le n :\, \sigma \in \mathcal{Y}} \left|\{i : i \in [i_\sigma, i_\sigma + l - 1] \cap [n], \mathsf{c}_i = \sigma\}\right|.$$

[3]This threshold is similar to the threshold used in [7].

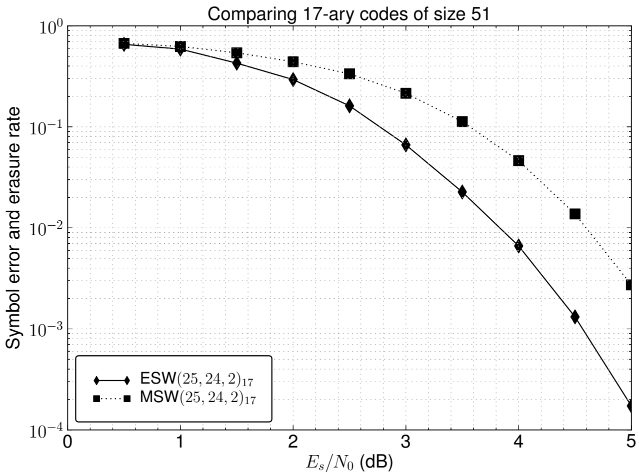

Fig. 5. Comparison of equitable and minimum symbol weight codes under cyclostationary noise.

The quantity $E(\mathcal{Y}; l, \mathsf{c})$ measures the maximum number of coordinates in c that can be affected by narrowband noise of duration l at symbols in $\mathcal{Y}$.

Let $L \subset \mathbb{Z}_{>0}$. We consider the following quantity as a function in e,

$$E(L, \mathcal{C}) : [q] \to [n],$$

$$E(e; L, \mathcal{C}) = \max_{\mathcal{Y} \subseteq \mathcal{X},\, |\mathcal{Y}|=e,\, l \in L,\, \mathsf{c} \in \mathcal{C}} E(\mathcal{Y}; l, \mathsf{c}),$$

then $E(e; L, \mathcal{C})$ measures the maximum number of coordinates, over all codewords in $\mathcal{C}$, that can be affected by e narrowband noise of duration $l \in L$. The following lemma states that it suffices to consider the maximum duration when determining the performance of a code in a PLC.

Lemma A.1. Let $\mathcal{C}$ be a q-ary code of length n. Consider $L \subset \mathbb{Z}_{>0}$ and define $n' = \min\{n, \max L\}$. Then

$$E(L, \mathcal{C}) = E(\{n'\}, \mathcal{C}).$$

Proof: Let $l' = \max L$ and fix $l \in L$ and $e \in [q]$.

Observe that since $[i, i+l-1] \subseteq [i, i+l'-1]$ for $i \le n$,

$$E(\mathcal{Y}; l, \mathsf{c}) \le E(\mathcal{Y}; l', \mathsf{c}) \text{ for } \mathsf{c} \in \mathcal{C},\, \mathcal{Y} \subset \mathcal{X}.$$

Hence, $E(e; \{l\}, \mathcal{C}) \le E(e; \{l'\}, \mathcal{C})$ and so, $E(e; L, \mathcal{C}) \le E(e; \{l'\}, \mathcal{C})$.

In addition, since $[i, i+l-1] \cap [n] \subseteq [n]$ for $i \le n$,

$$E(\mathcal{Y}; l, \mathsf{c}) \le E(\mathcal{Y}; n, \mathsf{c}) \text{ for } \mathsf{c} \in \mathcal{C},\, \mathcal{Y} \subset \mathcal{X}.$$

Similar argument shows that $E(e; L, \mathcal{C}) \le E(e; \{n\}, \mathcal{C})$. Since $l' \in L$, we have $E(e; L, \mathcal{C}) \ge E(e; \{l'\}, \mathcal{C})$ and the lemma follows. ■

The following is now immediate.

Corollary A.1. Let $\mathcal{C}$ be a q-ary code of length n. For $L \subset \mathbb{Z}_{>0}$,

$$E(e; L, \mathcal{C}) \le E(e; \{n\}, \mathcal{C}) \text{ for all } e \in [q].$$

Therefore, $E(L, \mathcal{C})$, which measures the maximum effect of narrowband noise on codewords, is maximized when $L = \{n\}$. Hence, we assume that only narrowband noise of duration n occurs.

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