Note

Graphical t-designs with block sizes three and four

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Abstract

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All graphical t-designs with $2 \leq t < k \leq 4$ are determined.

1. Introduction

A *t-wise balanced design* $T = (X, \mathcal{B})$ with parameters t-(v, K, λ) is a system $\mathcal{B}$ of subsets of size $k \in K$ (called *blocks*) from a set X of cardinality v such that each t-subset of X is contained in precisely λ blocks of $\mathcal{B}$. We assume that the blocks in $\mathcal{B}$ are not repeated. If $K = \{k\}$, we call T a *t-design* of type t-(v, k, λ).

Let $\mathcal{S}_Y$ denote the full symmetric group on the p-element set Y. Then $\mathcal{S}_Y$ acts in a natural way on the $\binom{p}{2}$ edges of the complete graph $K_p = (Y, \mathcal{E})$. A t-wise balanced design $T = (\mathcal{E}, \mathcal{B})$ for which $\mathcal{S}_Y$ is also an automorphism group is said to be *graphical*. It should be clear that if $B \in \mathcal{B}$, then B is a subgraph of K_p, and all subgraphs of K_p that are isomorphic to B are also in $\mathcal{B}$.

Chouinard, Kramer, and Kreher in [1] determined all graphical t-wise balanced designs with the restriction $\lambda \in \{1, 2\}$. All parameter situations for which there exists a graphical t-$(15, k, \lambda)$ design with $2 \leq t < k \leq 7$ were also determined by Kramer and Mesner [2]. In this note, we take a different direction and enumerate all graphical t-designs with the restriction $2 \leq t < k \leq 4$, but allowing arbitrary v's

and λ's. We note that it suffices to consider $0 < \lambda \leq \lceil \frac{1}{2}\binom{v-t}{k-t} \rceil$ since the complement of a t-(v, k, λ) design is a t-$(v, k, \binom{v-t}{k-t} - \lambda)$ design.

2. The enumeration

Let N_t denote the number of isomorphism classes of t-edge subgraphs of K_p. We define a *graphical extension matrix* G_{tk} to be an $N_t \times N_k$ matrix with rows indexed by the isomorphism classes $h_1^{(t)}, \ldots, h_{N_t}^{(t)}$ of t-edge subgraphs and columns indexed by the isomorphism classes $h_1^{(k)}, \ldots, h_{N_k}^{(k)}$ of k-edge subgraphs of K_p such that the (i, j)th entry of G_{tk} is the number of ways $h_i^{(t)}$ can be extended to a $h_j^{(k)}$. We remark that the G_{tk} matrix is exactly the same as the A_{tk} matrix as defined in [2] with $\mathcal{S}_Y$ acting on the t-subsets and k-subsets of $\mathscr{E}$. The following lemma is a special case of a more general result of Kramer and Mesner [2].

Lemma 1. *There exists a graphical t-(v, k, λ) design if and only if there is a (0, 1)-vector U satisfying*

$$G_{tk} U = \lambda J,$$

where J is the N_t-dimensional vector of all 1's.

There has been much effort and success in the design of efficient and effective heuristics for solving matrix equations such as $G_{tk} U = \lambda J$ when p is specified (see, for example, [4]). However, we have not come across any algorithm for the case when p is an unknown to be determined. Here, we propose to solve this problem by borrowing tools from symbolic computations. Solving $G_{tk} U = \lambda J$ is equivalent to finding subsets C of columns of G_{tk} whose row sum is uniform and equals λ. To do this, we generate subsets of columns and solve the N_t simultaneous diophantine equations involved. This task can be carried out with symbolic manipulation packages like MACSYMA or MAPLE. We can also restrict our search to $|C| \leq \lceil N_k/2 \rceil$ since the complement of a design is also a design. Based on this observation, we conducted an exhaustive search for all graphical 2-$(\binom{p}{2}, 3, \lambda)$, 2-$(\binom{p}{2}, 4, \lambda)$, and 3-$(\binom{p}{2}, 4, \lambda)$ designs which are presented in subsequent sections. The enumeration process using MAPLE took no more than a few minutes of CPU time on a VAX11-780 machine.

3. Data and results

In Fig. 1 is a list of the isomorphism classes of k-edge subgraphs of K_p for $2 \leq k \leq 4$. For simplicity of representation, we choose not to include isolated vertices in our figures. We also give in Fig. 2 the graphical extension matrices G_{23}, G_{24}, and G_{34} associated with the appropriate isomorphism classes. We have

$$k = 2$$

$$k = 3$$

$$k = 4$$

Fig. 1. Isomorphism classes of k-edge subgraphs, for $2 \le k \le 4$.

$$G_{23} = \begin{pmatrix} 0 & 2(p-3) & \binom{p-3}{2} & 1 & p-3 \\ \binom{p-4}{2} & 4 & 4(p-4) & 0 & 0 \end{pmatrix}$$

$$G_{24} = \begin{pmatrix} \binom{p-3}{2} & 5(p-3) & \binom{p-3}{2} & 8\binom{p-3}{2} & p-3 & 3\binom{p-3}{3} & 6\binom{p-3}{3} & 6\binom{p-3}{2} & 3\binom{p-3}{3} & 3\binom{p-3}{4} & 0 \\ 2(p-4) & 4 & 0 & 8(p-4) & 2 & 4\binom{p-4}{2} & 16\binom{p-4}{2} & 12(p-4) & 8\binom{p-4}{2} & 15\binom{p-4}{3} & 3\binom{p-4}{4} \end{pmatrix}$$

$$G_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 6(p-6) & \binom{p-6}{2} \\ 0 & 2 & 0 & 2(p-4) & 1 & 0 & \binom{p-4}{2} & 2(p-4) & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & p-5 & 2(p-5) & 4 & 2(p-5) & \binom{p-5}{2} & 0 \\ \binom{p-3}{2} & 3(p-3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & p-4 & 3(p-4) & 0 & \binom{p-4}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Fig. 2. Graphical extension matrices, G_{tk}, for $2 \leq t < k \leq 4$.

assumed that $p \geq 6$ in the G_{2k} matrices, and $p \geq 8$ in the G_{34} matrix. The graphical extension matrices for smaller values of p can be obtained by deleting the rows and columns corresponding to isomorphism classes of graphs that are on more than p vertices.

In the following subsections, we present all the graphical t-$(\binom{p}{2}, k, \lambda)$ designs for $2 \leq t < k \leq 4$. A t-design with block size k having blocks from the isomorphism classes $h_{i_1}^{(k)}, \ldots, h_{i_n}^{(k)}$ is represented by the set $\{i_1, \ldots, i_n\}$.

3.1. Graphical 2-designs with block size 3

The only graphical 2-$(\binom{p}{2}, 3, \lambda)$ designs are those listed in Table 1.

Table 1

Graphical 2-$(\binom{p}{2}, 3, \lambda)$ Designs			
$(\binom{p}{2}, \lambda)$	Representation	$(\binom{p}{2}, \lambda)$	Representation
(10,4)	{2}	(15,1)	{1,4}
(28,6)	{1,3,5}	(28,10)	{1,2}
(55,25)	{1,2,4,5}		

3.2. Graphical 2-designs with block size 4

The only graphical 2-$(\binom{p}{2}, 4, \lambda)$ designs are those listed in Table 2.

3.3. Graphical 3-designs with block size 4

The only graphical 3-$(\binom{p}{2}, 4, \lambda)$ design is the 3-(10, 4, 1) design and has representation $\{1, 3, 5\}$.

Table 2

Graphical 2-$(\binom{p}{2}, 4, \lambda)$ Designs			
$(\binom{p}{2}, \lambda)$	Representation	$(\binom{p}{2}, \lambda)$	Representation
(10,2)	{1,3}, {5}	(10,4)	{1,3,5}
(10,8)	{4}	(10,10)	{1,3,4}, {4,5}
(10,12)	{1,3,4,5}	(15,6)	{1,5}, {5,6}
(15,24)	{1,2,7}, {1,3,4,5,8,9}	(15,30)	{1,2,5,6,7}, {2,3,5,7,9}
(15,36)	{1,3,4,7}, {2,8,9}, {3,4,6,7}	(21,6)	{1}
(21,12)	{6}	(21,18)	{1,6}
(21,36)	{8}	(21,42)	{1,8}
(21,45)	{2,3,5,9,10}	(21,48)	{6,8}
(21,51)	{1,2,3,5,9,10}	(21,54)	{1,6,8}, {2,3,5,7}
(21,57)	{2,3,5,6,9,10}	(21,60)	{1,2,3,5,7}
(21,63)	{1,2,3,5,6,9,10}, {4,9,10}	(21,66)	{2,3,5,6,7}
(21,69)	{1,4,9,10}	(21,72)	{1,2,3,5,6,7}, {4,7}
(21,75)	{4,6,9,10}	(21,78)	{1,4,7}
(21,81)	{1,4,6,9,10}, {2,3,5,8,9,10}	(21,84)	{4,6,7}
(28,5)	{5,11}	(28,55)	{2,9,11}
(28,80)	{1,3,6,9}	(28,85)	{1,3,5,6,9,11}
(28,95)	{4,10,11}	(28,110)	{1,2,3,5,7}
(28,120)	{1,2,3,8,10}, {6,8,9}	(28,125)	{1,2,3,5,8,10,11}, {5,6,8,9,11}
(28,135)	{1,2,3,6,7,11}	(28,150)	{1,3,4,5,9,10}, {2,5,7,8}
(36,15)	{3,11}	(36,90)	{1,3,9}
(36,111)	{1,2,5,9,11}	(36,120)	{6,9}
(36,135)	{3,6,9,11}	(36,165)	{1,8,9,11}
(36,210)	{1,3,6,7}	(36,231)	{1,2,5,6,7,11}
(36,240)	{1,4,6,10}	(36,255)	{1,3,4,6,10,11}
(36,276)	{2,3,4,5,9,10}	(45,63)	{1,2,5,11}
(45,105)	{6,11}	(45,252)	{1,3,7}
(45,357)	{1,3,6,7,11}, {3,7,8,11}	(45,378)	{1,2,5,7,9}
(45,420)	{1,3,4,6,10}, {3,4,8,10}, {6,7,9}	(55,168)	{9}
(55,336)	{7}	(55,504)	{7,9}
(78,630)	{6,8,11}	(78,1080)	{1,3,7,8,11}
(78,1350)	{7,8,9,11}	(91,836)	{2,4,5,8,11}
(91,1430)	{4,7,11}	(91,1496)	{1,2,3,5,7,8,11}
(105,1320)	{1,3,4,6,11}	(105,1326)	{1,2,4,5,6,11}
(105,1650)	{3,4,8,9,11}	(105,1656)	{2,4,5,8,9,11}
(105,1782)	{3,6,8,9,11}	(105,1788)	{2,5,6,8,9,11}
(105,1980)	{1,3,4,7,11}	(105,1986)	{1,2,4,5,7,11}
(105,2112)	{1,3,6,7,11}	(105,2118)	{1,2,5,6,7,11}
(105,2442)	{3,7,8,9,11}	(105,2448)	{2,5,7,8,9,11}
(153,4935)	{4,6,7,11}	(153,5025)	{1,2,3,5,6,7,8,11}
(253,14535)	{10}		

4. Concluding remarks

In this note, we have demonstrated success in enumerating graphical *t*-designs for small values of *t* and *k* by a method that involves the use of symbolic computational tools. The major difficulty we face in attempting this method for higher values of *t* and *k* is that the number of subsets of the columns we have to consider in each case increases explosively. The polynomial entries of the graphical extension matrices also grow in complexity with *t* and *k*, and massive

amount of computing time will have to be invested to solve the simultaneous diophantine equations that arise.

It is also shown in this paper that for $k \in \{3, 4\}$, there is only a finite number of graphical t-(v, k, λ) designs. In relation to this, we mention the following conjecture of Chouinard [3].

Conjecture 1. For any fixed λ, there exist only finitely many graphical t-wise balanced designs.

This conjecture has been verified for $\lambda \in \{1, 2\}$ in [1], but remains open for all other λ's. A proof (or disproof) of this conjecture will be interesting.

References

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[2] E.S. Kramer and D.M. Mesner, t-Designs on hypergraphs, Discrete Math. 15 (1976) 263–296.
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[4] D.L. Kreher and S.P. Radziszowski, Finding simple t-designs by basis reduction, Congr. Numer. 55 (1986) 235–244.