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Note

# An improved finiteness theorem for graphical $t$-designs

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**Abstract**

We prove that there exist only finitely many nontrivial graphical $t$-$(v,k,\lambda)$ designs when $k \leqslant 4t/3$. This improves a previous result of Betten et al. (Discrete Math. 197/198 (1999) 83–109). 

We use the notation and terminology of [1] and assume that the reader is familiar with the concept of graphical $t$-designs [2]. All polynomials in this note are polynomials in $n$.

Betten et al. [1, Theorem 10] have shown that there exist only finitely many nontrivial graphical $t$-$(\binom{n}{2},k,\lambda)$ designs when $k=t+1$. In this note, we show that this finiteness result remains true when the condition $k=t+1$ is relaxed to $k \leqslant 4t/3$.

Let $t \geqslant 3$ and $k \leqslant 4t/3$. Let $I(t)$ denote the graph consisting of $t$ independent edges and define $\mathscr{K}$ to be the set of all graphs, each having $k$ edges and contains $I(t)$ as a subgraph. Then if $G \in \mathscr{K}$, $G$ must contain at least $t/3$ isolated edges.

By Alltop's Lemma (see [1, Lemma 2]), the entry in row $G$ and column $H$ of the polynomial Kramer–Mesner matrix is a polynomial whose degree is the difference in the sizes of the supports of $G$ and $H$. Hence, the entry in row $I(t)$ and column $I(k)$ of the polynomial Kramer–Mesner matrix is a polynomial of degree $2(k-t)$. The other entries in row $I(t)$ are polynomials of degree strictly less than $2(k-t)$.

Without loss of generality, assume that $I(k)$ is a block of a graphical $t$-$(\binom{n}{2},k,\lambda)$ design $\mathscr{D}$. The columns indexed by graphs in $\mathscr{K}\setminus\{I(k)\}$ each has an entry a polynomial of degree $2(k-t)$, precisely in the row indexed by the graph obtained by removing $k-t \leqslant t/3$ isolated edges from the graph indexing the corresponding column. Hence for large $n$, all graphs in $\mathscr{K}\setminus\{I(k)\}$ must also be blocks of $\mathscr{D}$. This forces $\mathscr{D}$ to be the complete design and establishes the following result.

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**Theorem 1.** *There exist only finitely many nontrivial graphical* $t$-$(\binom{n}{2}, k, \lambda)$ *designs when* $k \leqslant 4t/3$.

## References

[1] A. Betten, M. Klin, R. Laue, A. Wassermann, Graphical $t$-designs vai polynomial Kramer–Mesner matrices, Discrete Math. 197/198 (1999) 83–109.

[2] Y.M. Chee, Graphical designs, in: C.J. Colbourn, J.H. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, FL, 1996, pp. 366–369 (Chapter. IV.23).