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Highly Symmetric Expanders

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Expander graphs are relevant to theoretical computer science in addition to the construction of high-performance switching networks. In communication network applications, a high degree of symmetry in the underlying topology is often advantageous, as it may reduce the complexity of designing and analyzing switching and routing algorithms. We give explicit constructions of expander graphs that are highly symmetric. In particular, we construct infinite families of Ramanujan graphs with large guarantees on the orders of their automorphism groups. Although nonlinear, our expander graphs are within a constant factor of the size of the smallest graphs exhibiting the same expansion properties. This work generalizes and extends in several directions a previous explicit construction of expander graphs based on finite projective spaces due to Alon. © 2002 Elsevier Science (USA)

1. INTRODUCTION

Informally, a graph is an expander if every subset of vertices has a surprisingly large neighborhood. Expanders are used in the construction of telephone networks [9], nonblocking networks [13], superconcentrators [26], virtual circuits [12], sorting and selection algorithms [2, 4], and graphs that are hard to pebble [20, 24], as well as in obtaining lower bounds [31, 32], in

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the establishment of time–space trade-offs [1,17,30], and in reducing the amount of randomness required in probabilistic computations [16,27]. Expanders have also found applications in the design of linear-time encodable–decodable error-correcting codes with high rates [28].

It is not too difficult to prove the existence of expanders using probabilistic methods. However, for applications, an explicit construction of an expander is desirable. Naively, one could construct expanders by sampling a graph at random and then checking the sampled graph to see if it is an expander. However, the problem of verifying if a given graph is an expander has been shown to be co $\mathcal{NP}$-complete [10].

Another drawback with the method of random sampling is that the generated graph is likely to have a low degree of symmetry (we measure the symmetry of a graph here by the order of its full automorphism group), since almost all graphs have no nontrivial automorphism groups (see [11]). For applications especially in the design of telecommunication networks, it is often desirable to have expanders with high symmetry. In this paper, we study the problem of explicitly constructing expanders with large automorphism groups. Our constructions employ algebraic and geometric techniques.

2. PRELIMINARIES

Given a graph $\Gamma = (V, E)$ and $S \subset V$, the *neighborhood* of S is the set $N(S) = \{v \mid \{u, v\} \in E \text{ for some } u \in S\}$. Let $\Gamma = (V, E)$ be a k-regular bipartite graph with bipartition $V = I \dot{\cup} O$ such that Γ is *balanced*; that is, $|I| = |O| = n$. The vertices in I and O are called *inputs* and *outputs*, respectively. The expansion property basically means that every subset of inputs must have many outputs in its neighborhood. The definition for expander graphs is as follows: Γ is said to have *expansion* c if c is the largest value so that for every subset $S \subseteq I$,

$$|N(S)| \geq \left(1 + c\left(1 - \frac{|S|}{n}\right)\right)|S|. \tag{1}$$

The expansion of Γ is denoted expan(Γ). A bipartite graph with n inputs and n outputs that is k-regular is called an *(n, k, c)-expander* if it has expansion c.

Expansions of graphs are closely related to the eigenvalues of certain associated matrices. Given a graph $\Gamma = (V, E)$, we can define a $|V| \times |V|$ matrix $\mathfrak{A}$ with rows and columns indexed by vertices in V such that

$$\mathfrak{A}_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E; \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $\mathfrak{A}$ is called the *A-matrix* of Γ. If, in addition, Γ is a balanced bipartite graph, we can associate another $|V|/2 \times |V|/2$ matrix $\mathfrak{B}$ whose rows are indexed by inputs and columns indexed by outputs such that

$$\mathfrak{B}_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E; \\ 0 & \text{otherwise.} \end{cases}$$

This matrix $\mathfrak{B}$ is called the *B-matrix* of Γ. Observe that if $\mathfrak{B}$ is defined, we have

$$\mathfrak{A} = \begin{bmatrix} 0 & \mathfrak{B} \\ \mathfrak{B}^T & 0 \end{bmatrix}. \tag{2}$$

The eigenvalues of $\mathfrak{A}$ are denoted $\lambda_1, \ldots, \lambda_{2n}$, where $\lambda_1 \geq \cdots \geq \lambda_{2n}$. The eigenvalues of the matrix $\mathfrak{B}\mathfrak{B}^T$ are denoted $\mu_1, \ldots, \mu_n$, where $\mu_1 \geq \cdots \geq \mu_n$. We refer to these eigenvalues as eigenvalues of Γ. Note that since $\mathfrak{B}\mathfrak{B}^T$ is positive semidefinite, the eigenvalues μ_i are all nonnegative. From (2), one can also conclude that μ_i is an eigenvalue of $\mathfrak{B}\mathfrak{B}^T$ if and only if $\pm\sqrt{\mu_i}$ are both eigenvalues of $\mathfrak{A}$.

A *Ramanujan graph* [21] is a k-regular graph whose eigenvalue λ_2 satisfies

$$\lambda_2 \leq 2\sqrt{k-1}.$$

An *automorphism* of a bipartite graph $\Gamma = (V, E)$ is a bijection π: $V \to V$, mapping inputs to inputs and outputs to outputs, such that $\{u, v\} \in E$ if and only if $\{\pi(u), \pi(v)\} \in E$. The set of all automorphisms of Γ forms a group under functional composition. We call this group the *full automorphism group* of Γ and denote it by Aut(Γ). Any subgroup of Aut(Γ) is simply called an automorphism group of Γ.

3. PREVIOUS AND PRESENT WORK

Explicit constructions for expanders have been given by Margulis [22], Gabber and Galil [14], Tanner [29], and Alon [4]. Tanner [29] was the first to show the relation of eigenvalues with the expansion property. In particular, he obtained the following result.

LEMMA 3.1 (Tanner). *Let $\Gamma = (I \dot{\cup} O, E)$ be a k-regular balanced bipartite graph, then for any $S \subset I$, we have*

$$|N(S)| \geq \frac{k^2|S|}{(k^2 - \mu_2)|S|/|I| + \mu_2}. \tag{3}$$

COROLLARY 3.1 *$expan(\Gamma) \geq 1 - \mu_2/k^2$.*

Proof. From (1) and (3), we derive

$$\operatorname{expan}(\Gamma) \geq \inf_{0 \leq \alpha \leq 1} f(\alpha),$$

where

$$f(\alpha) = \frac{1}{1-\alpha}\left(\frac{k^2}{(k^2-\mu_2)\alpha + \mu_2} - 1\right).$$

Simple calculus shows that $\operatorname{expan}(\Gamma) \geq \lim_{\alpha\to 1} f(\alpha) = 1 - \mu_2/k^2$. ∎

Lemma 3.1 and its corollary are useful for deriving expansion properties. In fact, most of the work on expanders after Gabber and Galil [14] hinges on bounding μ_2. The observation that the smaller μ_2 is, the better expansion we can derive led to the consideration of Ramanujan graphs. The importance of the upper bound $2\sqrt{k-1}$ lies in the following lower bound of Alon and Boppana (see [3]). Let $(\Gamma_{n,k})_{n \geq 1}$ be a family of k-regular graphs on n vertices. Then

$$\liminf_{n\to\infty} \lambda_2(\Gamma_{n,k}) \geq 2\sqrt{k-1}.$$

Hence the Ramanujan graphs make good expanders. The construction of Ramanujan graphs by Lubotzky *et al.* [21] and independently by Margulis [23] is one of the major developments in constructive methods.

While it is desirable to have expanders with large automorphism groups for applications to networks, the automorphism groups of expanders have not been widely studied, except for a result of Klawe [18] and another of Alon and Roichman [5].

The expanders that Alon constructed in [4] are point–hyperplane incidence graphs of the finite projective spaces $PG(t, q)$, $q = p^e$ a prime power, and hence has the group $P\Gamma L(t+1, \mathbb{F}_q)$ of order $eq^{\binom{t+1}{2}}\prod_{i=2}^{t+1}(q^i - 1)$ as the full automorphism group (see [6]). In this paper, we generalize the construction of Alon [4] based on finite projective spaces in several directions. In particular, we construct Ramanujan graphs with large guarantees on the order of their automorphism groups. Among all graphs with the same expansion properties, our graphs have the optimal number of edges (up to a constant factor).

4. THE CONSTRUCTION

Let $\mathbb{F}_q$ be the finite field with q elements, and denote by $\mathbb{F}_q^*$ the group of all nonzero elements in $\mathbb{F}_q$. Define S_t to be the set of all nonzero $(t+1)$-tuples

$(a_0, \ldots, a_t)$ of elements of $\mathbb{F}_q$:

$$S_t = \mathbb{F}_q^{t+1} \backslash \{(0, \ldots, 0)\}.$$

The finite projective space $PG(t, q)$ may be defined to be an incidence structure $\mathscr{S} = (\mathscr{P}, \mathscr{H}, \mathscr{I})$ such that both $\mathscr{P}$ and $\mathscr{H}$ are the sets of equivalence classes of elements of S_t under the equivalence relation given by

$$(a_0, \ldots, a_t) \sim (b_0, \ldots, b_t)$$

if and only if there exists $r \in \mathbb{F}_q^*$ such that $a_i = rb_i$ for all $i \in \{0, \ldots, t\}$.

We denote a point $P \in \mathscr{P}$ by $(a_0 : \cdots : a_t)$ if $(a_0, \ldots, a_t)$ lies in the equivalence class P, and we denote a hyperplane $H \in \mathscr{H}$ by $[x_0 : \cdots : x_t]$ if $(x_0, \ldots, x_t)$ lies in the equivalence class H. The incidence relation $\mathscr{I}$ is defined as follows: $(P, H) \in \mathscr{I}$ if and only if

$$\sum_{i=0}^{t} a_i x_i = 0 \tag{4}$$

Let ζ be a positive integer. Let $\mathscr{P}_\zeta = \{P_\zeta^{(i)} | 1 \leq i \leq \zeta, \quad P \in \mathscr{P}\}$, $\mathscr{H}_\zeta = \{H_\zeta^{(j)} | 1 \leq j \leq \zeta, H \in \mathscr{H}\}$, and $\mathscr{I}_\zeta = \{(P_\zeta^{(i)}, H_\zeta^{(j)}) | 1 \leq i \leq \zeta, 1 \leq j \leq \zeta$, and $(P, H) \in \mathscr{I}\}$. We call $\mathscr{S}_\zeta = (\mathscr{P}_\zeta, \mathscr{H}_\zeta, \mathscr{I}_\zeta)$ the *ζ-blowup* of $\mathscr{S}$. Let $P \in \mathscr{P}$ and $H \in \mathscr{H}$. The set $\{P_\zeta^{(i)} | 1 \leq i \leq \zeta\}$ is called the *fiber of* P, and the set $\{H_\zeta^{(j)} | 1 \leq j \leq \zeta\}$ is the *fiber of* H. We define $\Psi_\zeta : \mathscr{S}_\zeta \to \mathscr{S}$ to be the map such that $\Psi_\zeta(P_\zeta^{(i)}) = P$ for any $P_\zeta^{(i)} \in \mathscr{P}_\zeta$, and $\Psi_\zeta \ (H_\zeta^{(j)}) = H$ for any $H_\zeta^{(j)} \in \mathscr{H}_\zeta$.

Let I_ζ and O_ζ be sets of vertices corresponding to the elements of $\mathscr{P}_\zeta$ and $\mathscr{H}_\zeta$, respectively. In I_ζ, we denote the elements by $(a_0 : \cdots : \ a_t)_\zeta^{(i)}$, and in O_ζ, we denote the elements by $[x_0 : \cdots : x_t]_\zeta^{(j)}$.

Definition 4.1. $\Gamma_\zeta(t, q) = (I_\zeta \dot{\cup} O_\zeta, E)$ is the balanced bipartite graph such that for $\mathfrak{a} = (a_0 : \cdots : a_t)_\zeta^{(i)} \in I_\zeta$ and $\mathfrak{x} = [x_0 : \cdots : x_t]_\zeta^{(j)} \in O_\zeta$, we have $\{\mathfrak{a}, \mathfrak{x}\} \in E$ if and only if (4) is satisfied.

We adopt the convention that if ζ is not specified, then ζ is taken to be one. The graphs $\Gamma(t, q)$ are the point–hyperplane incidence graphs of the finite projective spaces $PG(t, q)$ and hence are isomorphic to the expanders constructed by Alon in [4].

Let I' and O' be sets of vertices corresponding to the elements of S_t. In I', we denote the elements by $(a_0, \ldots, a_t)$, and in O', we denote the elements by $[x_0, \ldots, x_t]$. (Note that I' and O' are essentially punctured affine spaces.)

Definition 4.2. $\Gamma'(t, q) = (I' \cup O', E)$ is the balanced bipartite graph such that for $\mathfrak{a} = (a_0, \ldots, a_t) \in I'$ and $\mathfrak{x} = [x_0, \ldots, x_t] \in O'$, we have $\{\mathfrak{a}, \mathfrak{x}\} \in E$ if and only if

$$\sum_{i=0}^{t} a_i x_i = 1. \tag{5}$$

We remark that the definitions of inputs and outputs in $\Gamma_\zeta(t, q)$ and $\Gamma'(t, q)$ satisfy the principle of duality.

5. COMBINATORIAL PROPERTIES

In this section, we give some combinatorial properties of the graphs $\Gamma_\zeta(t, q)$ and $\Gamma'(t, q)$ defined in the previous section.

5.1. Number and Degree of Vertices

PROPOSITION 5.1. $|I_\zeta| = |O_\zeta| = \zeta(q^{t+1} - 1)/(q - 1)$ *and the degree of each vertex in* $\Gamma_\zeta(t, q)$ *is* $\zeta(q^t - 1)/(q - 1)$.

Proof. Obvious from definition. ■

To compute the degree of vertices in $\Gamma'(t, q)$ we require the following lemma.

LEMMA 5.1. *Let* $a_0, \ldots, a_t \in \mathbb{F}_q$. *The equation*

$$\sum_{i=0}^{t} a_i x_i = 1$$

has q^t *solutions* $(x_0, \ldots, x_t) \in \mathbb{F}_q^{t+1}$ *except when* $a_0 = \cdots = a_t = 0$, *in which case no solutions exist.*

Proof. It is obvious that no solutions exist when $a_0 = \cdots = a_t = 0$. Suppose that not all a_i's are zero. We may assume without loss of generality that $a_t \neq 0$. Then we get

$$x_t = a_t^{-1}\left(1 - \sum_{i=0}^{t-1} a_i x_i\right).$$

Since there are no restrictions on the choices of $x_0, \ldots, x_{t-1}$, it follows that there are q^t solutions. ■

COROLLARY 5.1. *The degree of each vertex in* $\Gamma'(t, q)$ *is* q^t.

5.2. Strong Neighborhood Properties

In this section, we determine the number of outputs that are adjacent to both of two given inputs in our graphs $\Gamma_\zeta(t, q)$ and $\Gamma'(t, q)$.

PROPOSITION 5.2. *Let* $\mathfrak{a} = (a_0 : \cdots : a_t)_\zeta^{(i)}$ *and* $\mathfrak{b} = (b_0 : \cdots : b_t)_\zeta^{(j)}$ *be any two inputs in* $\Gamma_\zeta(t, q)$. *Then the number of outputs adjacent to both* $\mathfrak{a}$ *and* $\mathfrak{b}$ *is*

$$\zeta(\tfrac{q^t-1}{q-1}), \qquad \textit{if } \Psi_\zeta(\mathfrak{a}) = \Psi_\zeta(\mathfrak{b});$$

$$\zeta(\tfrac{q^{t-1}-1}{q-1}), \qquad \textit{if } \Psi_\zeta(\mathfrak{a}) \neq \Psi_\zeta(\mathfrak{b}).$$

Proof. If $\mathfrak{a}$ and $\mathfrak{b}$ lie in the same fiber, then an output $\mathfrak{x} = [x_0 : \cdots : x_t]_\zeta^{(k)}$ is adjacent to $(a_0 : \cdots : a_t)_\zeta^{(i)}$ if and only if it is adjacent to $(b_0 : \cdots : b_t)_\zeta^{(j)}$. Therefore, the number of outputs adjacent to both $\mathfrak{a}$ and $\mathfrak{b}$ is the number of outputs adjacent to any one of them. This number is given by Proposition 5.1.

If $\mathfrak{a}$ and $\mathfrak{b}$ lie in different fibers, then $\Psi_\zeta(\mathfrak{a})$ and $\Psi_\zeta(\mathfrak{b})$ are distinct inputs in $\Gamma(t, q)$. The number of outputs adjacent to both $\Psi_\zeta(\mathfrak{a})$ and $\Psi_\zeta(\mathfrak{b})$ in $\Gamma(t, q)$ is $(q^{t-1} - 1)/(q - 1)$, so the number of outputs adjacent to both $\mathfrak{a}$ and $\mathfrak{b}$ in $\Gamma_\zeta(t, q)$ is $\zeta(q^{t-1} - 1)/(q - 1)$. ■

PROPOSITION 5.3. *Let* $\mathfrak{a} = (a_0, \ldots, a_t)$ *and* $\mathfrak{b} = (b_0, \ldots, b_t)$ *be two distinct inputs of* $\Gamma'(t, q)$. *Assume that* $a_k \neq 0$. *Then the number of outputs* $\mathfrak{x} = [x_0, \ldots, x_t]$ *adjacent to both* $\mathfrak{a}$ *and* $\mathfrak{b}$ *is*

$$\begin{array}{ll} 0, & \textit{if } a_k b_i - a_i b_k = 0 \textit{ for all } i \in \{0, 1, \ldots, t\}; \\ q^{t-1}, & \textit{otherwise.} \end{array}$$

Proof. The number of outputs $\mathfrak{x}$ adjacent to both $\mathfrak{a}$ and $\mathfrak{b}$ is the number of solutions to the simultaneous equations

$$\sum_{i=0}^{t} a_i x_i = 1 \qquad \text{and} \qquad \sum_{i=0}^{t} b_i x_i = 1. \tag{6}$$

For simplicity of notation, and without loss of generality, we may and do assume $a_0 \neq 0$ (i.e., $k = 0$) in the following. Eliminating x_0 from (6), we get

$$\sum_{i=1}^{t} (a_0 b_i - a_i b_0) x_i = a_0 - b_0. \tag{7}$$

We distinguish the cases $a_0 = b_0$ and $a_0 \neq b_0$.

When $a_0 = b_0$, one of $a_0 b_i - a_i b_0$, $i \in \{1, \ldots, t\}$, must be nonzero. Therefore, the number of $(x_1, \ldots, x_t)$ satisfying (7) is q^{t-1}. From (6), the value of

x_0 is uniquely determined by the t-tuple $(x_1, \dots, x_t)$. Hence, the number of nonzero $(x_0, \dots, x_t)$ is q^{t-1}.

When $a_0 \neq b_0$, then by Lemma 5.1, the number of $(x_1, \dots, x_t)$ is 0 if $a_0 b_i - a_i b_0 = 0$ for all $i \in \{0, 1, \dots, t\}$ and q^{t-1} otherwise.

Since x_0 is uniquely determined by $(x_1, \dots, x_t)$, the number of nonzero $(x_0, \dots, x_t)$ found is as required. ■

PROPOSITION 5.4. *Given any input $\mathfrak{a}$ of $\Gamma_\zeta(t, q)$, there exist t distinct outputs $\mathfrak{x}_1, \dots, \mathfrak{x}_t$ adjacent to $\mathfrak{a}$ such that there are exactly ζ inputs, each of which is adjacent to all of $\mathfrak{x}_1, \dots, \mathfrak{x}_t$.*

Proof. If suffices to show that in $\Gamma(t, q)$, the outputs $\Psi_\zeta(\mathfrak{x}_1), \dots, \Psi_\zeta(\mathfrak{x}_t)$ have exactly one input (that is, $\Psi_\zeta(\mathfrak{a})$) adjacent to all of them. Since $\Gamma(t, q)$ is the point–hyperplane incidence graph of the projective space $PG(t, q)$, the existence and uniqueness of such an input are guaranteed. ■

6. COMPUTING EIGENVALUES

In this section, we compute the eigenvalues $\mu_1, \dots, \mu_n$ of the B-matrices of $\Gamma_\zeta(t, q)$ and $\Gamma'(t, q)$.

THEOREM 6.1. *Let $\mathfrak{B}$ be the B-matrix of $\Gamma_\zeta(t, q)$. Then the eigenvalues of $\mathfrak{B}\mathfrak{B}^T$ are:*

Eigenvalue	*Multiplicity*
$\zeta^2 \left(\frac{q^t - 1}{q - 1}\right)^2$	1
$\zeta^2 q^{t-1}$	$q \left(\frac{q^t - 1}{q - 1}\right)$
0	$(\zeta - 1) \left(\frac{q^{t+1} - 1}{q - 1}\right)$

Proof. Label the inputs of $\Gamma_\zeta(t, q)$ by $\mathfrak{a}_1, \dots, \mathfrak{a}_n$, $n = \zeta(q^{t+1} - 1)/(q - 1)$. Then it is easy to see that $(\mathfrak{B}\mathfrak{B}^T)_{ij}$ is the number of outputs adjacent to both $\mathfrak{a}_i$ and $\mathfrak{a}_j$. This number is given by Proposition 5.2. Let $\mathfrak{J}$ be the $\zeta \times \zeta$ matrix

$$\begin{bmatrix} \zeta & \cdots & \zeta \\ \vdots & \ddots & \vdots \\ \zeta & \cdots & \zeta \end{bmatrix}.$$

Arranging the inputs in such a way that for each $i = 1, \dots, (q^{t+1} - 1)/(q - 1)$, the inputs $\mathfrak{a}_{(i-1)\zeta + 1}, \dots, \mathfrak{a}_{i\zeta}$ belong to the same fiber under Ψ_ζ, it follows that

$$\mathfrak{B}\mathfrak{B}^T = \begin{bmatrix} \mathfrak{D}_{11} & \cdots & \mathfrak{D}_{1,\frac{q^{t+1}-1}{q-1}} \\ \vdots & \ddots & \vdots \\ \mathfrak{D}_{\frac{q^{t+1}-1}{q-1},1} & \cdots & \mathfrak{D}_{\frac{q^{t+1}-1}{q-1},\frac{q^{t+1}-1}{q-1}} \end{bmatrix},$$

where

$$\mathfrak{D}_{ij} = \begin{cases} (\frac{q^t-1}{q-1})\,\mathfrak{J} & \text{if } i=j; \\ (\frac{q^{t-1}-1}{q-1})\ \mathfrak{J} & \text{if } i \neq j. \end{cases}$$

In other words,

$$\mathfrak{B}\mathfrak{B}^T = \mathfrak{M} \otimes \mathfrak{J},$$

where $\mathfrak{M}$ is the B-matrix of $\Gamma(t, q)$. Now, the eigenvalues of $\mathfrak{M}$ and the corresponding multiplicities are well known. They are:

Eigenvalue	Multiplicity
$(\frac{q^t-1}{q-1})^2$	1
q^{t-1}	$q\,(\frac{q^t-1}{q-1})$

It is an easy exercise in linear algebra to see the eigenvalues of $\mathfrak{J}$ and their corresponding multiplicities are:

Eigenvalue	Multiplicity
ζ^2	1
0	$(\zeta-1)$

The eigenvalues of $\mathfrak{M} \otimes \mathfrak{J}$ are all the possible products $\lambda\mu$, where λ is an eigenvalue of $\mathfrak{M}$ and μ is an eigenvalue of $\mathfrak{J}$, counted with multiplicities. Theorem 6.1 follows. ■

COROLLARY 6.1. *For $\zeta \leq 4$, $\Gamma_\zeta(t, q)$ is a Ramanujan graph for all prime powers q.*

COROLLARY 6.2. *$\Gamma_\zeta(t, q)$ is an (n, k, c)-expander, where $n = \zeta(q^{t+1}-1)/(q-1)$, $k = \zeta(q^t-1)/(q-1)$, and $c \geq 1 - 1/q^{t-1}$.*

It is actually easy to see that the analysis of expansion properties carried out by Alon [4] on the graph $\Gamma(t, q)$ based on Lemma 3.1 implies the same expansion properties for our graphs $\Gamma_\zeta(t, q)$. To see this, we note that the number of vertices in $\Gamma_\zeta(t, q)$ and the degree of each vertex in $\Gamma_\zeta(t, q)$ are precisely ζ times the corresponding values in $\Gamma(t, q)$. Furthermore, the second largest eigenvalue of the B-matrix of $\Gamma_\zeta(t, q)$ is ζ^2 times the second largest eigenvalue of the B-matrix of $\Gamma(t, q)$. Hence, ζ vanishes when the various parameters of $\Gamma_\zeta(t, q)$ are substituted into the inequality in Lemma 3.1. The result below now follows from Theorem 2.3 in [4].

THEOREM 6.2. *For every subset S of inputs in $\Gamma_\zeta(t, q)$ we have*

$$|N(S)| \geq n - \frac{n^{1+1/t}}{|S|},$$

where $n = \zeta(q^{t+1} - 1)/(q - 1)$ is the number of inputs in $\Gamma_\zeta(t, q)$.

We now determine the eigenvalues of $\Gamma'(t, q)$.

THEOREM 6.3. *Let $\mathfrak{B}$ be the B-matrix of $\Gamma'(t, q)$. Then the eigenvalues of $\mathfrak{B}\mathfrak{B}^T$ are:*

Eigenvalue	*Multiplicity*
q^{2t}	1
q^t	$(q-2)(\frac{q^{t+1}-1}{q-1})$
q^{t-1}	$q(\frac{q^t-1}{q-1})$

Proof. Label the inputs of $\Gamma'(t, q)$ by $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$, $n = q^{t+1} - 1$. Then $(\mathfrak{B}\mathfrak{B}^T)_{ij}$ is the number of outputs adjacent to both $\mathfrak{a}_i$ and $\mathfrak{a}_j$. It follows from Corollary 5.1 and Proposition 5.3 that this number is given by

(i) q^t, if $\mathfrak{a}_i = \mathfrak{a}_j$;
(ii) 0, if $\mathfrak{a}_i \neq \mathfrak{a}_j$ and $\Psi_{q-1}(\mathfrak{a}_i) = \Psi_{q-1}(\mathfrak{a}_j)$;
(iii) q^{t-1}, otherwise.

Given $\mathfrak{a}_i$, the number of $\mathfrak{a}_j$ satisfying (ii) is $q - 2$ and the number of $\mathfrak{a}_j$ satisfying (iii) is $q^{t+1} - q$. We label the inputs in such a way that for each $i = 1, \ldots, (q^{t+1} - 1)/(q - 1)$, the inputs $\mathfrak{a}_{(i-1)(q-1)+1}, \ldots, \mathfrak{a}_{i(q-1)}$ satisfy (i) or (ii) pairwise. Then

$$\mathfrak{B}\mathfrak{B}^T = \begin{bmatrix} \mathfrak{D}_{11} & \cdots & \mathfrak{D}_{1,\frac{q^{t+1}-1}{q-1}} \\ \vdots & \ddots & \vdots \\ \mathfrak{D}_{\frac{q^{t+1}-1}{q-1},1} & \cdots & \mathfrak{D}_{\frac{q^{t+1}-1}{q-1},\frac{q^{t+1}-1}{q-1}} \end{bmatrix},$$

where each $\mathfrak{D}_{ij}$ is a $(q-1)\times(q-1)$ matrix such that

$$\mathfrak{D}_{ij} = \begin{cases} \begin{bmatrix} q^t & 0 & \cdots & 0 \\ 0 & q^t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q^t \end{bmatrix} = q^t I, & \text{if } i = j; \\ \begin{bmatrix} q^{t-1} & \cdots & q^{t-1} \\ \vdots & \ddots & \vdots \\ q^{t-1} & \cdots & q^{t-1} \end{bmatrix}, & \text{if } i \neq j. \end{cases}$$

Now we consider the eigenvalues and eigenvectors of $\mathfrak{B}\mathfrak{B}^T$.

The vectors $\mathfrak{v}_1 = [1, \dots, 1]^t$ is clearly an eigenvector with eigenvalue $q^t + (q(q^t-1)/(q-1))(q-1)q^{t-1} = q^{2t}$. So the eigenspace of q^{2t} has dimension at least one.

For $i = 2, \dots, (q^{t+1}-1)/(q-1)$, let $\mathfrak{v}_i = (v_{ij})_j$ be the vector whose entries are

$$v_{ij} = \begin{cases} 1, & \text{for } 1 \leq j \leq q-1; \\ -1, & \text{for } (i-1)(q-1)+1 \leq j \leq i(q-1); \\ 0, & \text{otherwise.} \end{cases}$$

These are clearly eigenvectors with eigenvalues $q^t + (q-1)(-q^{t-1}) = q^{t-1}$. It is equally clear that $\mathfrak{v}_2, \dots, \mathfrak{v}_{(q^{t+1}-1)/(q-1)}$ are linearly independent. So the eigenspace of q^{t-1} has dimension at least $q(q^t-1)/(q-1)$.

For each $i = 1, \dots, (q^{t+1}-1)/(q-1)$ and each $j = 2, \dots, q-1$, let $\mathfrak{w}_{ij} = (w_{ijk})_k$ be the vector whose entries are

$$w_{ijk} = \begin{cases} 1, & \text{if } k = (i-1)(q-1)+1; \\ -1, & \text{if } k = (i-1)(q-1)+j; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, all these vectors are linearly independent, and they are all eigenvectors with eigenvalue q^t. Consequently, the eigenspace of q^t has dimension at least $(q^{t+1}-1)(q-2)/(q-1)$.

Since $1 + q(q^t-1)/(q-1) + (q^{t+1}-1)(q-2)/(q-1) = q^{t+1}-1$, it follows that the inequalities involving the dimensions of the eigenspaces above are actually all equalities. This completes the proof. ■

COROLLARY 6.3. *$\Gamma'(t, q)$ is a Ramanujan graph for all prime powers q.*

Corollary 6.4. *$\Gamma'(t, q)$ is an (n, k, c)-expander, where $n = q^{t+1} - 1$, $k = q^t$, and $c \geq 1 - 1/q^t$.*

Theorem 6.4. *For every subset S of inputs in $\Gamma'(t, q)$, we have*

$$|N(S\ )| \geq n - \frac{n^{(t+2)/(t+1)}}{|S|},$$

where $n = q^{t+1} - 1$ is the number of inputs in $\Gamma'(t, q)$.

Proof. We know $n = q^{t+1} - 1$, $k = q^t$ and $\mu_2 = q^t$. Let $\alpha = |S|/n$. By Lemma 3.1, we have

$$\begin{aligned}|N(S)| &\geq \frac{\alpha q^t n}{\alpha q^t + 1 - \alpha}\\ &= n - \frac{(1-\alpha)(q^{t+1}-1)}{\alpha q^t + 1 - \alpha}\\ &\geq n - \left(\frac{1-\alpha}{\alpha}\right)q\\ &= n - \frac{n^{(t+2)/(t+1)}(1 + 1/n)^{1/(t+1)}}{|S|} - \frac{|S|(n+1)^{1/(t+1)}}{|S|}\\ &\geq n - \frac{n^{(t+2)/(t+1)}}{|S|}. \qquad \blacksquare\end{aligned}$$

7. AUTOMORPHISM GROUPS

In the previous section, we showed that for $\zeta \leq 4$, the graphs $\Gamma_\zeta(t, q)$ and $\Gamma'(t, q)$ are Ramanujan graphs with good expansions. We now show that these graphs are highly symmetric as well. First, we determine the order of the full automorphism group of $\Gamma_\zeta(t, q)$.

Lemma 7.1. *Every automorphism α of $\Gamma_\zeta(t, q)$ induces an automorphism $\bar{\alpha}$ of $\Gamma(t, q)$.*

Proof. Let α be an automorphism of $\Gamma_\zeta(t, q)$. It suffices to show that if α sends an input $\mathfrak{a}$ of $\Gamma_\zeta(t, q)$ to $\mathfrak{a}'$, then the whole fiber of $\mathfrak{a}$ under Ψ_ζ is sent to the fiber of $\mathfrak{a}'$ under Ψ_ζ. (By duality, the analogous result holds when the inputs are replaced by outputs of $\Gamma_\zeta(t, q)$.)

By Proposition 5.4, a given input $\mathfrak{a}$ of $\Gamma_\zeta(t, q)$ has at least t distinct outputs $\mathfrak{x}_1, \dots, \mathfrak{x}_t$ adjacent to $\mathfrak{a}$ such that there are exactly ζ inputs which make up the fiber containing $\mathfrak{a}$ under Ψ_ζ, each of which is adjacent to all $\mathfrak{x}_1, \dots, \mathfrak{x}_t$.

Consequently, the images $\alpha(\mathfrak{x}_1), \dots, \alpha(\mathfrak{x}_t)$ are outputs in $\Gamma_\zeta(t, q)$ adjacent to $\mathfrak{a}' = \alpha(\mathfrak{a})$, and $\alpha(\mathfrak{x}_1), \dots, \alpha(\mathfrak{x}_t)$ have precisely ζ inputs, each adjacent to all of them, that constitute the fiber under Ψ_ζ containing $\mathfrak{a}'$. Hence, the fiber of $\mathfrak{a}$ under Ψ_ζ is sent by α to the fiber of $\mathfrak{a}'$ under Ψ_ζ. ■.

THEOREM 7.1. *There is a group K of order $(\zeta!)^{2n}$ (where $n = (q^{t+1} - 1)/(q - 1)$) such that the sequence*

$$1 \to K \to \mathrm{Aut}(\Gamma_\zeta(t, q)) \xrightarrow{\beta} \mathrm{Aut}(\Gamma(t, q)) \to 1$$

is exact, where $\beta(\alpha) = \bar{\alpha}$. In particular, the order of

$$\mathrm{Aut}(\Gamma_\zeta(t, q)) \text{ is } (\zeta!)^{2n} |\, P\Gamma L(t + 1, \mathbb{F}_q)|.$$

Proof. Let γ be an automorphism of $\Gamma(t, q)$. For an input $\mathfrak{a}$ of $\Gamma(t, q)$, let $\mathfrak{a}' = \gamma(\mathfrak{a})$. For an output $\mathfrak{x}$ of $\Gamma(t, q)$, let $\mathfrak{x}' = \gamma(\mathfrak{x})$. Suppose further that the fiber of $\mathfrak{a}$ (respectively, $\mathfrak{x}$) under Ψ_ζ (in $\Gamma_\zeta(t, q)$) is $\{\mathfrak{a}_1, \dots, \mathfrak{a}_\zeta\}$ (respectively, $\{\mathfrak{x}_1, \dots, \mathfrak{x}_\zeta\}$), and similarly for $\mathfrak{a}'$ and $\mathfrak{x}'$.

Let α be a map from $\Gamma_\zeta(t, q)$ to itself defined as follows: α is a bijection from $\{\mathfrak{a}_1, \dots, \mathfrak{a}_\zeta\}$ to $\{\mathfrak{a}'_1, \dots, \mathfrak{a}'_\zeta\}$ for all inputs $\mathfrak{a}$ of $\Gamma(t, q)$; α is a bijection from $\{\mathfrak{x}_1, \dots, \mathfrak{x}_\zeta\}$ to $\{\mathfrak{x}'_1, \dots, \mathfrak{x}'_\zeta\}$ for all outputs $\mathfrak{x}$ of $\Gamma(t, q)$. If $\mathfrak{a}_i$ and $\mathfrak{x}_j$ are adjacent, then $\mathfrak{a}$ and $\mathfrak{x}$ are adjacent in $\Gamma(t, q)$, so $\mathfrak{a}'$ and $\mathfrak{x}'$ are adjacent and hence $\alpha(\mathfrak{a}_i)$ and $\alpha(\mathfrak{x}_j)$ are incident. Hence, α define an automorphism of $\Gamma_\zeta(t, q)$ and $\bar{\alpha} = \gamma$. This proves the surjectivity of β.

It remains to show that $K \overset{\text{def}}{=} \ker\beta$ is of order $(\zeta!)^{2n}$. For this, we note that $\alpha \in K$ implies that for an input $\mathfrak{a}$ of $\Gamma_\zeta(t, q)$ (respectively, an output $\mathfrak{x}$ of $\Gamma_\zeta(t, q)$), $\alpha(\mathfrak{a})$ (respectively, $\alpha(\mathfrak{x})$) is in the fiber under Ψ_ζ containing $\mathfrak{a}$ (respectively, $\mathfrak{x}$). Therefore, the automorphism α fixes each fiber but can freely permute all the inputs (respectively, outputs) in each fiber. Conversely, if α is a map that simply permutes all the inputs (respectively, ouputs) in each fiber, then α clearly gives rise to an automorphism of $\Gamma_\zeta(t, q)$ and $\bar{\alpha}$ is the identity automorphism of $\Gamma(t, q)$. The number of such automorphisms α is clearly $(\zeta!)^{2n}$. ■

For $\Gamma'(t, q)$, we are not able to determine the precise order of its full automorphism group. However, we can determine a rather large automorphism group of $\Gamma'(t, q)$.

LEMMA 7.2. *Elements of $GL(t + 1, \mathbb{F}_q)$ induce automorphisms of* $\Gamma'(t, q)$.

Proof. Let α be an element of $GL(t + 1, \mathbb{F}_q)$. An input $\mathfrak{a}$ of $\Gamma'(t, q)$ may be written as $\begin{bmatrix} a_0 \\ \vdots \\ a_t \end{bmatrix}$ with $a_0, \dots, a_t \in \mathbb{F}_q$. Let $\bar{\alpha}(a) = \alpha \begin{bmatrix} a_0 \\ \vdots \\ a_t \end{bmatrix}$. An output $\mathfrak{x}$ of $\Gamma'(t, q)$

may be written as $[x_0, \dots, x_t]$ with $x_0, \dots, x_t \in \mathbb{F}_q$. Let $\bar{\alpha}(\mathfrak{x})$ be the output given by $[x_0, \dots, x_t]\alpha^{-1}$. Then α induces a bijection $\bar{\alpha}$ on the set of all inputs as well as the set of all outputs of $\Gamma'(t, q)$. Furthermore, if $\mathfrak{a}$ and $\mathfrak{x}$ are incident, that is,

$$[x_0, \dots, x_t]\begin{bmatrix} a_0 \\ \vdots \\ a_t \end{bmatrix} = \sum_{i=0}^{t} a_i x_i = 1, \tag{8}$$

then

$$[x_0, \dots, x_t]\, \alpha^{-1}\alpha\begin{bmatrix} a_0 \\ \vdots \\ a_t \end{bmatrix} = \sum_{i=0}^{t} a_i x_i = 1,$$

implying that $\bar{\alpha}(\mathfrak{a})$ and $\bar{\alpha}(\mathfrak{x})$ are adjacent. ■

LEMMA 7.3. *Let* $\mathfrak{a} = \begin{bmatrix} a_0 \\ \vdots \\ a_t \end{bmatrix}$ *be an input of* $\Gamma'(t, q)$ *and let* $\mathfrak{x} = [x_0, \dots, x_t]$ *be an output of* $\Gamma'(t, q)$. *Let* σ *be an automorphism of* $\mathbb{F}_q$ *that sends* $f \in \mathbb{F}_q$ *to* $f^\sigma \in \mathbb{F}_q$. *Then* σ *induces an automorphism* $\bar{\sigma}$ *of* $\Gamma'(t, q)$ *given by*

$$\bar{\sigma}(\mathfrak{a}) = \begin{bmatrix} a_0^\sigma \\ \vdots \\ a_t^\sigma \end{bmatrix}$$

$$\bar{\sigma}(\mathfrak{x}) = [x_0^\sigma, \dots, x_t^\sigma].$$

Proof. It is clear that $\bar{\alpha}$ is a bijection on the set of inputs as well as the set of outputs of $\Gamma'(t, q)$. If $\mathfrak{a}$ and $\mathfrak{x}$ are adjacent, applying σ to (8) shows that

$$\sum_{i=0}^{t} a_i^\sigma x_i^\sigma = 1;$$

that is, $\bar{\alpha}(\mathfrak{a})$ and $\bar{\alpha}(\mathfrak{x})$ are adjacent. ■

LEMMA 7.4. *Let* $\bar{\alpha}$ *and* $\overline{\beta}$ *be automorphisms of* $\Gamma'(t, q)$ *induced by* α, $\beta \in GL(t + 1, \mathbb{F}_q)$. *Let* $\bar{\sigma}$ *and* $\bar{\tau}$ *be automorphisms of* $\Gamma'(t, q)$ *induced by* σ, $\tau \in \operatorname{Aut} \mathbb{F}_q$. *If* $\bar{\sigma}\bar{\alpha} = \bar{\tau}\overline{\beta}$, *then* $\bar{\alpha} = \overline{\beta}$ *and* $\bar{\sigma} = \bar{\tau}$.

Proof. If $\bar{\sigma}\bar{\alpha} = \bar{\tau}\overline{\beta}$, then $\bar{\tau}^{-1}\bar{\sigma} = \overline{\beta}\bar{\alpha}^{-1}$. Clearly, $\bar{\tau}^{-1}\bar{\sigma}$ fixes each vector in the standard basis of $\mathbb{F}_q^{t+1}$. Hence, $\overline{\beta}\bar{\alpha}^{-1}$ is the identity automorphism, as the only element of $GL(t + 1, \mathbb{F}_q)$ fixing each vector in the standard basis of $\mathbb{F}_q^{t+1}$ is the identity. Hence, $\bar{\alpha} = \overline{\beta}$ and $\bar{\sigma} = \bar{\tau}$. ■

THEOREM 7.2. $\mathrm{Aut}(\Gamma'(t, q))$ *contains a subgroup* G *that satisfies the exact sequence*

$$1 \to GL(t+1, \mathbb{F}_q) \to G \to \mathrm{Aut}\,\mathbb{F}_q \to 1.$$

In particular, G *has order*

$$|\mathrm{Aut}\,\mathbb{F}_q| \cdot q^{\binom{t+1}{2}} \prod_{i=1}^{t+1} (q^i - 1).$$

Proof. Let G be the set

$$G = \{\bar{\sigma}\bar{\alpha} \mid \sigma \in \mathrm{Aut}\,\mathbb{F}_q,\ \alpha \in GL(t+1, \mathbb{F}_q)\}.$$

If $\alpha = (\alpha_{ij}) \in GL(t+1, \mathbb{F}_q)$, then for $\tau \in \mathrm{Aut}\,\mathbb{F}_q$, $\bar{\alpha}\bar{\tau} = \bar{\tau}\bar{\alpha}'$, where $\alpha' = (\alpha_{ij}^{\tau^{-1}})$. Hence, G is a subgroup of $\mathrm{Aut}(\Gamma'(t, q))$ since, given $\bar{\sigma}\bar{\alpha}$, $\bar{\tau}\bar{\beta} \in G$, we have

$$(\bar{\sigma}\bar{\alpha})(\bar{\tau}\bar{\beta})^{-1} = \bar{\sigma}\bar{\alpha}\bar{\beta}^{-1}\bar{\tau}^{-1} = \overline{\sigma\tau^{-1}(\alpha\beta^{-1})'} \in G.$$

Let π: $G \to \mathrm{Aut}\,\mathbb{F}_q$ be the map $\pi(\bar{\sigma}\bar{\alpha}) = \bar{\sigma}$. Then π is clearly a well-defined, surjective group homomorphism whose kernel is $GL(t+1, \mathbb{F}_q)$. ■

8. SOME REMARKS ON DENSITY

One of the main interests in expanders is to construct low-density expanders, or expanders with few edges. The graphs $\Gamma_\zeta(t, q)$ and $\Gamma'(t, q)$ we constructed are expanders with $(\zeta^{1/t} + o(1))n^{2-1/t}$ and $(1 + o(1))n^{2-1/(t+1)}$ edges, respectively, where n is the number of inputs. As pointed out by Alon [4], the well-known results on the problem of Zarankiewicz [15] can be used to prove the following result.

PROPOSITION 8.1 (Alon). *If* $\Gamma = (I \dot{\cup} O, E)$ *is a balanced bipartite graph with* $|I| = |O| = n$ *such that for all* $S \subset I$ *we have*

$$|N(S)| \geq n - \frac{n^{1+1/t}}{|S|}$$

then $|E| \geq \Omega(n^{2-1/t})$.

This result, together with Theorems 6.2 and 6.4, implies that for ζ a fixed constant, our graphs $\Gamma_\zeta(t, q)$ and $\Gamma'(t, q)$ are all highly expanding graphs with the optimal number of edges (up to a constant factor). The case $\zeta = 1$ was obtained by Alon [4].

9. CONCLUSION

Generalizing the work of Alon, we have given explicit constructions of two infinite families of highly expanding graphs, $\Gamma_\zeta(t, q)$ and $\Gamma'(t, q)$. These graphs are Ramanujan graphs for $\zeta \leq 4$ and are shown to have large automorphism groups. It is also shown that these graphs contain the fewest possible number of edges (up to a constant factor) among all graphs with the same expansion properties.

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