An Enumeration of Graphical Designs

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Abstract: **Let $\Psi(t, k)$ denote the set of pairs (v, λ) for which there exists a graphical t-(v, k, λ) design. Most results on graphical designs have gone to show the finiteness of $\Psi(t, k)$ when t and k satisfy certain conditions. The exact determination of $\Psi(t, k)$ for specified t and k is a hard problem and only $\Psi(2, 3)$, $\Psi(2, 4)$, $\Psi(3, 4)$, $\Psi(4, 5)$, and $\Psi(5, 6)$ have been determined. In this article, we determine completely the sets $\Psi(2, 5)$ and $\Psi(3, 5)$. As a result, we find more than 270,000 inequivalent graphical designs, and more than 8,000 new parameter sets for which there exists a graphical design. Prior to this, graphical designs are known for only 574 parameter sets.**
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1. INTRODUCTION

For a finite set X and a nonnegative integer t, the set of all t-subsets of X is denoted $\binom{X}{t}$. A *k-uniform set system* is a pair $(X, \mathcal{B})$, where X is a finite set of elements called *points* and $\mathcal{B} \subseteq \binom{X}{k}$. Elements of $\mathcal{B}$ are called *blocks*. The *order* of $(X, \mathcal{B})$ is the number of points, $|X|$. A *design* with parameters t-(v, k, λ) is a k-uniform set system $(X, \mathcal{B})$ of order v such that every $T \in \binom{X}{t}$ is contained in exactly λ blocks of $\mathcal{B}$. To avoid triviality, we impose the following restrictions on a t-(v, k, λ) design $(X, \mathcal{B})$:

1. $t \geq 2$,

2. $t < k$,
3. $\mathcal{B} \neq \emptyset$, and $\mathcal{B} \neq \binom{X}{k}$.

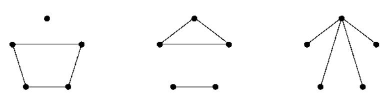
For two designs, $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, an *isomorphism* of $(X, \mathcal{A})$ onto $(Y, \mathcal{B})$ is a bijection $\sigma : X \to Y$ such that $\sigma(\mathcal{A}) = \mathcal{B}$. An *automorphism* of a design is an isomorphism of the design onto itself. The set of all automorphisms of a design $\mathcal{D}$ forms a group under functional composition. This group is called the *automorphism group* of $\mathcal{D}$ and is denoted by $\mathrm{Aut}(\mathcal{D})$. A subgroup $H \leq \mathrm{Aut}(\mathcal{D})$ is a *group of automorphisms* of $\mathcal{D}$.

Let V be a set of cardinality n and consider the induced action of the symmetric group $\mathcal{S}_n = \mathrm{Sym}(V)$ on the set $X = \binom{V}{2}$. This defines an embedding of $\mathcal{S}_n$ into $\mathcal{S}_{\binom{n}{2}} = \mathrm{Sym}(X)$ with image group $\mathcal{S}_n^{[2]}$. By canonical extension, $\mathcal{S}_n^{[2]}$ also acts on $\binom{X}{k}$. A t-(v, k, λ) design $(X, \mathcal{B})$ is *graphical* if it has a group of automorphisms that is permutation isomorphic to $\mathcal{S}_n^{[2]}$ with $v = \binom{n}{2}$. In particular, $\mathcal{B}$ is then a union of orbits of $\mathcal{S}_n^{[2]}$ on $\binom{X}{k}$.

The term "graphical design" is motivated by the following alternative perspective. Considering the complete graph K_n with vertex set V, we may view X as the edge set of K_n, in which case the orbits of $\mathcal{S}_n^{[2]}$ on $\binom{X}{k}$ are in a one-to-one correspondence with the isomorphism classes of spanning k-edge subgraphs of K_n. Thus, we may view the block set $\mathcal{B}$ of a graphical design as a set of spanning k-edge subgraphs of K_n, closed under isomorphism of graphs, such that every t-edge subgraph of K_n is a subgraph of λ graphs in $\mathcal{B}$. Although the definition of a graphical design does not explicitly assume this graphical structure, a required group of automorphisms induces the structure (in a canonical manner for $n \neq 4$) because one of the orbits of $\mathcal{S}_n^{[2]}$ corresponds to the line graph of K_n, from which one can recover the sets of edges having a vertex in common when $n \neq 4$. Two graphical designs $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, with individualized required groups of automorphisms, H and K, respectively, are *equivalent* if there exists an isomorphism σ of $(X, \mathcal{A})$ onto $(Y, \mathcal{B})$ such that $\sigma H \sigma^{-1} = K$.

The first example of a graphical design has been attributed to R. M. Wilson by Kramer and Mesner [12]:

Example 1.1. *A graphical* 3-*(10, 4, 1) design is obtained by taking as blocks all spanning* 4-*edge subgraphs of* K_5 *isomorphic to one of the following graphs:*



However, Betten et al. [1] have reported that already in 1970, M.H. Klin has described graphical designs when he determined the overgroups of $\mathcal{S}_n^{[2]}$. But Klin's result was unpublished, except for a short note that appeared in a less well-known journal [10]. Further examples of graphical designs were given by Driessen [9]. The first systematic approach to determining the existence of graphical designs was undertaken by Chouinard et al. [8], who determined all graphical t-(v, k, λ) designs with $\lambda = 1$ and $\lambda = 2$. These results led Chouinard [6] to make the following conjecture, which remains open.

Conjecture 1.2 (Chouinard). *For any fixed* λ*, there exist only finitely many graphical* t-(v, k, λ) *designs.*

Partial progress on this conjecture has been obtained by Chouinard [7].

Computers were brought to bear in the early nineties, which resulted in further progress in the construction of graphical t-(v, k, λ) designs. Kreher et al. [13] used the LLL algorithm to construct many examples of graphical t-(v, k, λ) designs. Chee [2,3] used symbolic computational methods to find all graphical 2-(v, 3, λ), 2-(v, 4, λ), 3-(v, 4, λ), and 4-(v, 5, λ) designs. Further sporadic examples were also obtained by Kramer [11] and Chee [4]. In the late nineties, more graphical t-(v, k, λ) designs were discovered by Betten et al. [1] using an improved implementation of the LLL algorithm. This is the state-of-the-art. Despite that more than 20 years have passed since the introduction of graphical designs, only a small finite number of them are known. Let $\Lambda(t, k, v)$ denote the set of $\lambda \le \frac{1}{2}\binom{v-t}{k-t}$ for which a graphical t-(v, k, λ) design exists, and let $\Psi(t, k) = \{(v, \lambda) : \lambda \in \Lambda(t, k, v)\}$. The reason for restricting $\lambda \le \frac{1}{2}\binom{v-t}{k-t}$ is to avoid duplication by complementation, since if $(X, \mathcal{B})$ is a (graphical) t-(v, k, λ) design, then its *complement*, $(X, \binom{X}{k} \setminus \mathcal{B})$, is a (graphical) t-(v, k, $\binom{v-t}{k-t} - \lambda$) design. The parameters of all graphical designs known are given in Appendix A, where Table I presents those sets $\Psi(t, k)$ which we have complete knowledge of, and Table III lists known elements of some $\Psi(t, k)$ which we have yet to completely determine. The authority for these tables are [1–4,11,13] (cf. [5]). In total, there are only 574 parameter sets for which we know there exist graphical designs. Indeed, results in the literature are either on construction of sporadic examples, on nonexistence, or on the finiteness of the number of graphical designs with certain parameters.

The purpose of this article is to improve this state of knowledge by determining completely the sets $\Psi(2, 5)$ and $\Psi(3, 5)$. With this result, the sets $\Psi(t, k)$ are now completely known for $2 \le t < k \le 5$. As a by-product, we give more than 8,000 new parameter sets for which there exists a graphical design, substantially improving on the number of graphical designs known thus far. Our results also correct some minor errors in [1].

2. KRAMER–MESNER MATRICES AND OUTLINE OF APPROACH

Suppose we wish to construct a t-(v, k, λ) design $(X, \mathcal{B})$ with a group of automorphisms Γ. Then $\mathcal{B}$ is a union of orbits of Γ on $\binom{X}{k}$. Let $\mathcal{O}_1^{(t)}, \mathcal{O}_2^{(t)}, \ldots, \mathcal{O}_{N(t)}^{(t)}$ and $\mathcal{O}_1^{(k)}, \mathcal{O}_2^{(k)}, \ldots, \mathcal{O}_{N(k)}^{(k)}$ be the orbits of Γ on $\binom{X}{t}$ and on $\binom{X}{k}$, respectively. Define an $N(t) \times N(k)$ integer matrix $W_{t,k}(X|\Gamma)$ by the rule that the (i, j)-entry is $|\{K \in \mathcal{O}_j^{(k)} : K \supseteq T\}|$, where $T \in \mathcal{O}_i^{(t)}$ can be chosen arbitrarily. Such $W_{t,k}(X|\Gamma)$ matrices are called *Kramer–Mesner matrices*, after Kramer and Mesner [12] who observed the following.

Theorem 2.1 (Kramer and Mesner)**.** *There exists a* t-(v, k, λ) *design with a group of automorphisms* Γ *if and only if there exists a* $\{0, 1\}$*-vector* u *such that*

$$W_{t,k}(X|\Gamma)\mathsf{u} = \lambda(1, \ldots, 1)^{\mathsf{T}}. \tag{1}$$

Based on Theorem 2.1, our approach to determining $\Psi(2, 5)$ and $\Psi(3, 5)$ is to find all solutions to the equation $W_{t,k}(X|\mathcal{S}_n^{[2]})\mathsf{u} = \lambda(1, \ldots, 1)^{\mathsf{T}}$ for $(t, k) = (2, 5)$ and $(t, k) = (3, 5)$. More precisely, we perform the following steps:

(i) determine a bound n_0 so that no graphical t-(v, k, λ) design exists for $n \ge n_0$; and
(ii) enumerate all graphical t-($\binom{n}{2}$, k, λ) designs for $n < n_0$ by determining all solutions to $W_{t,k}(X|\mathcal{S}_n^{[2]})\mathsf{u} = \lambda(1, \ldots, 1)^{\mathsf{T}}$.

The first step is accomplished via a combinatorial analysis and the second step is accomplished via computation. It is not hard to see that distinct $\{0, 1\}$-vectors u satisfying (1) give inequivalent graphical designs.

$$
\begin{array}{c}
\left[\begin{array}{cc}
4(n-3) & 4 \\
\frac{5}{2}(n-3)^{\underline{3}} & 10(n-4)^{\underline{2}} \\
6(n-3)^{\underline{2}} & 16(n-4) \\
7(n-3)^{\underline{2}} & 12(n-4) \\
4(n-3)^{\underline{2}} & 4(n-4) \\
\frac{2}{3}(n-3)^{\underline{3}} & 4(n-4)^{\underline{2}} \\
\frac{1}{8}(n-3)^{\underline{4}} & \frac{7}{6}(n-4)^{\underline{3}} \\
\frac{2}{3}(n-3)^{\underline{4}} & 4(n-4)^{\underline{3}} \\
5(n-3)^{\underline{3}} & 20(n-4)^{\underline{2}} \\
\frac{3}{2}(n-3)^{\underline{3}} & 4(n-4)^{\underline{2}} \\
\frac{3}{2}(n-3)^{\underline{4}} & 14(n-4)^{\underline{3}} \\
4(n-3)^{\underline{3}} & 24(n-4)^{\underline{2}} \\
\frac{1}{4}(n-3)^{\underline{5}} & 4(n-4)^{\underline{4}} \\
6(n-3)^{\underline{2}} & 16(n-4) \\
(n-3)^{\underline{2}} & 4(n-4) \\
\frac{3}{2}(n-3)^{\underline{4}} & 14(n-4)^{\underline{3}} \\
5(n-3)^{\underline{3}} & 20(n-4)^{\underline{2}} \\
2(n-3)^{\underline{4}} & 12(n-4)^{\underline{3}} \\
\frac{7}{3}(n-3)^{\underline{3}} & 4(n-4)^{\underline{2}} \\
\frac{1}{48}(n-3)^{\underline{6}} & \frac{3}{4}(n-4)^{\underline{5}} \\
\frac{1}{4}(n-3)^{\underline{5}} & 4(n-4)^{\underline{4}} \\
\frac{1}{8}(n-3)^{\underline{5}} & \frac{7}{6}(n-4)^{\underline{4}} \\
\frac{1}{2}(n-3)^{\underline{3}} & 3(n-4)^{\underline{2}} \\
\frac{1}{4}(n-3)^{\underline{4}} & \frac{2}{3}(n-4)^{\underline{3}} \\
\frac{1}{6}(n-3)^{\underline{3}} & 0 \\
0 & \frac{1}{48}(n-4)^{\underline{6}}
\end{array}\right]
\end{array}
\begin{array}{c}
\text{index} \\
1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \\ 25 \\ 26
\end{array}
$$

FIGURE 1. Transpose of the Kramer–Mesner matrix $W_{2,5}(X|\mathcal{S}_n^{[2]})$.

Betten et al. [1] have computed the matrices $W_{2,5}(X|\mathcal{S}_n^{[2]})$ and $W_{3,5}(X|\mathcal{S}_n^{[2]})$. These take the forms given in Figures 1 and 2, where $n^{\underline{k}}$ denotes the *falling factorial* $n(n-1)\cdots(n-k+1)$. Observe that the matrices have constant row sum $\binom{\binom{n}{2}-t}{k-t}$. A list of orbit representatives indexing the rows and columns of $W_{2,5}(X|\mathcal{S}_n^{[2]})$ and $W_{3,5}(X|\mathcal{S}_n^{[2]})$ is given in Appendix B.

3. UPPER BOUNDS FOR EXISTENCE

Our subsequent proofs of the nonexistence of graphical designs for n large enough in the cases $(t, k) = (2, 5)$ and $(t, k) = (3, 5)$ are quantitative versions of the proof of a finiteness theorem of Betten et al. [1].

The orbit of a graph G under the action of $\mathcal{S}_n^{[2]}$ is denoted by $\mathrm{Orb}(G)$.

					index
$3(n-3)$	0	3	3	0	1
$\frac{3}{2}(n-3)^{\underline{3}}$	$5(n-5)$	$(n-4)^{\underline{2}}$	$\frac{3}{2}(n-4)^{\underline{2}}$	12	2
$3(n-3)^{\underline{2}}$	8	$4(n-4)$	$3(n-4)$	0	3
$3(n-3)^{\underline{2}}$	4	$5(n-4)$	$6(n-4)$	0	4
$\frac{3}{2}(n-3)^{\underline{2}}$	1	$2(n-4)$	$6(n-4)$	0	5
$\frac{1}{2}(n-3)^{\underline{3}}$	$3(n-5)$	0	0	0	6
$\frac{1}{8}(n-3)^{\underline{4}}$	$\frac{1}{2}(n-5)^{\underline{2}}$	0	0	$3(n-6)$	7
0	$3(n-5)^{\underline{2}}$	0	$\frac{1}{2}(n-4)^{\underline{3}}$	0	8
0	$12(n-5)$	$3(n-4)^{\underline{2}}$	$3(n-4)^{\underline{2}}$	0	9
0	$2(n-5)$	$(n-4)^{\underline{2}}$	$\frac{3}{2}(n-4)^{\underline{2}}$	0	10
0	$7(n-5)^{\underline{2}}$	$\frac{1}{2}(n-4)^{\underline{3}}$	0	$24(n-6)$	11
0	$12(n-5)$	$3(n-4)^{\underline{2}}$	0	24	12
0	$\frac{3}{2}(n-5)^{\underline{3}}$	0	0	$12(n-6)^{\underline{2}}$	13
0	6	$6(n-4)$	$3(n-4)$	0	14
0	2	$n-4$	0	0	15
0	$5(n-5)^{\underline{2}}$	$(n-4)^{\underline{3}}$	0	$36(n-6)$	16
0	$8(n-5)$	$4(n-4)^{\underline{2}}$	$3(n-4)^{\underline{2}}$	24	17
0	$5(n-5)^{\underline{2}}$	$(n-4)^{\underline{3}}$	$\frac{3}{2}(n-4)^{\underline{3}}$	$24(n-6)$	18
0	$2(n-5)$	$(n-4)^{\underline{2}}$	$4(n-4)^{\underline{2}}$	0	19
0	$\frac{1}{8}(n-5)^{\underline{4}}$	0	0	$\frac{7}{2}(n-6)^{\underline{3}}$	20
0	$(n-5)^{\underline{3}}$	$\frac{1}{8}(n-4)^{\underline{4}}$	0	$15(n-6)^{\underline{2}}$	21
0	$\frac{1}{2}(n-5)^{\underline{3}}$	0	$\frac{1}{8}(n-4)^{\underline{4}}$	$3(n-6)^{\underline{2}}$	22
0	$n-5$	$\frac{1}{2}(n-4)^{\underline{2}}$	0	6	23
0	$\frac{1}{2}(n-5)^{\underline{2}}$	0	$\frac{1}{2}(n-4)^{\underline{3}}$	0	24
0	0	0	$\frac{1}{2}(n-4)^{\underline{2}}$	0	25
0	0	0	0	$\frac{1}{8}(n-6)^{\underline{4}}$	26

FIGURE 2. Transpose of the Kramer–Mesner matrix $W_{3,5}(X|\mathcal{S}_n^{[2]})$.

A. Upper Bound for Existence of Graphical 2-$(v, 5, \lambda)$ Designs

We prove in this section that no graphical 2-$(\binom{n}{2}, 5, \lambda)$ design exists if $n \geq 538$.

Let $(X, \mathcal{B})$ be a graphical 2-$(\binom{n}{2}, 5, \lambda)$ design, where $n \geq 538$. We may assume without loss of generality that $\mathcal{B} \supseteq Orb(G_{26}^{(5)})$, since otherwise we can consider the complement of the design. Let μ_i denote the sum of all entries of degree i (as a polynomial in n) in row two of $W_{2,5}(X|\mathcal{S}_n^{[2]})$. Then we have $\mu_6 = (n-4)^{\underline{6}}/48$, $\mu_5 = 3(n-4)^{\underline{5}}/4$, $\mu_4 = 55(n-4)^{\underline{4}}/6$, $\mu_3 = 275(n-4)^{\underline{3}}/6$, and $\mu_2 = 89(n-4)^{\underline{2}}$. Define the integers $\lambda_6 = \mu_6$ and $\lambda_i = \lambda_{i+1} + \mu_i$ for $i = 2, 3, 4, 5$. By considering the number of blocks in $Orb(G_{26}^{(5)})$ containing $G_2^{(2)}$, we see that

$$\lambda \geq \lambda_6. \tag{2}$$

Lemma 3.1. $\mathcal{B} \supseteq Orb(G_{20}^{(5)})$.

Proof. Suppose that $\mathcal{B} \not\supseteq Orb(G_{20}^{(5)})$. Then by considering the number of blocks in $\mathcal{B}$ containing $G_1^{(2)}$, we have

$$\lambda \leq \binom{\binom{n}{2} - 2}{3} - \frac{1}{48}(n-3)^{\underline{6}}.$$

The above inequality, together with inequality (2), implies

$$\lambda_6 \leq \binom{\binom{n}{2} - 2}{3} - \frac{1}{48}(n-3)^{\underline{6}},$$

giving

$$n^6 - 69n^5 + 1085n^4 - 8435n^3 + 36642n^2 - 84664n + 80832 \leq 0,$$

which is impossible for $n \geq 51$. □

So $\mathcal{B} \supseteq \bigcup_{i \in \{20,26\}} Orb(G_i^{(5)})$ and by considering the number of blocks in $\mathcal{B}$ containing $G_2^{(2)}$, we now have

$$\lambda \geq \lambda_5. \tag{3}$$

Lemma 3.2. $\mathcal{B} \supseteq \bigcup_{i \in \{13,21,22\}} Orb(G_i^{(5)})$.

Proof. Suppose that $\mathcal{B}$ contains at most two of the orbits $Orb(G_i^{(5)})$, $i \in \{13, 21, 22\}$. Then by considering the number of blocks in $\mathcal{B}$ containing $G_1^{(2)}$, we have

$$\lambda \leq \binom{\binom{n}{2} - 2}{3} - \frac{1}{8}(n-3)^{\underline{5}}.$$

The above inequality, together with inequality (3), implies

$$\lambda_5 \leq \binom{\binom{n}{2} - 2}{3} - \frac{1}{8}(n-3)^{\underline{5}},$$

giving

$$3n^5 - 295n^4 + 4475n^3 - 28541n^2 + 85198n - 98184 \leq 0,$$

which is impossible for $n \geq 82$. □

So $\mathcal{B} \supseteq \bigcup_{i \in \{13,20,21,22,26\}} Orb(G_i^{(5)})$ and by considering the number of blocks in $\mathcal{B}$ containing $G_2^{(2)}$, we now have

$$\lambda \geq \lambda_4. \tag{4}$$

Lemma 3.3. $\mathcal{B} \supseteq \bigcup_{i\in\{7,8,11,16,18,24\}} Orb(G_i^{(5)})$.

Proof. Suppose that $\mathcal{B}$ contains at most five of the orbits $Orb(G_i^{(5)})$, $i \in \{7, 8, 11, 16, 18, 24\}$. Then by considering the number of blocks in $\mathcal{B}$ containing $G_1^{(2)}$, we have

$$\lambda \le \binom{\binom{n}{2}-2}{3} - \frac{1}{8}(n-3)^{\underline{4}}.$$

The above inequality, together with inequality (4), implies

$$\lambda_4 \le \binom{\binom{n}{2}-2}{3} - \frac{1}{8}(n-3)^{\underline{4}},$$

giving

$$3n^4 - 1154n^3 + 14721n^2 - 64450n + 95256 \le 0,$$

which is impossible for $n \ge 372$. □

So $\mathcal{B} \supseteq \bigcup_{i\in\{7,8,11,13,16,18,20,21,22,24,26\}} Orb(G_i^{(5)})$ and by considering the number of blocks in $\mathcal{B}$ containing $G_2^{(2)}$, we now have

$$\lambda \ge \lambda_3. \tag{5}$$

Lemma 3.4. $\mathcal{B} \supseteq \bigcup_{i\in\{2,6,9,10,12,17,19,23,25\}} Orb(G_i^{(5)})$.

Proof. Suppose that $\mathcal{B}$ contains at most eight of the orbits $Orb(G_i^{(5)})$, $i \in \{2, 6, 9, 10, 12, 17, 19, 23, 25\}$. Then by considering the number of blocks in $\mathcal{B}$ containing $G_1^{(2)}$, we have

$$\lambda \le \binom{\binom{n}{2}-2}{3} - \frac{1}{6}(n-3)^{\underline{3}}.$$

The above inequality, together with inequality (5), implies

$$\lambda_3 \le \binom{\binom{n}{2}-2}{3} - \frac{1}{6}(n-3)^{\underline{3}},$$

giving

$$n^3 - 546n^2 + 4541n - 9516 \le 0,$$

which is impossible for $n \ge 538$. □

So $\mathcal{B} \supseteq \bigcup_{i\in\{2,6,7,8,9,10,11,12,13,16,17,18,19,20,21,22,23,24,25,26\}} Orb(G_i^{(5)})$ and by considering the number of blocks in $\mathcal{B}$ containing $G_2^{(2)}$, we now have

$$\lambda \geq \lambda_2. \tag{6}$$

Lemma 3.5. $\mathcal{B} \supseteq \bigcup_{i\in\{3,4,5,14,15\}} Orb(G_i^{(5)})$.

Proof. Suppose that $\mathcal{B}$ contains at most four of the orbits $Orb(G_i^{(5)})$, $i \in \{3, 4, 5, 14, 15\}$. Then by considering the number of blocks in $\mathcal{B}$ containing $G_1^{(2)}$, we have

$$\lambda \leq \binom{\binom{n}{2}-2}{3} - (n-3)^{\underline{2}}.$$

The above inequality, together with inequality (6), implies

$$\lambda_2 \leq \binom{\binom{n}{2}-2}{3} - (n-3)^{\underline{2}},$$

giving

$$n^2 - 59n + 216 \leq 0,$$

which is impossible for $n \geq 56$. □

So $\mathcal{B} \supseteq \bigcup_{i\in\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26\}} Orb(G_i^{(5)}) = \binom{X}{5} \setminus Orb(G_1^{(5)})$. If $\mathcal{B} \not\supseteq Orb(G_1^{(5)})$, then $(X, \mathcal{B})$ cannot be a 2-$(\binom{n}{2}, 5, \lambda)$ design unless $4(n-3) = 4$, which is impossible for $n \geq 5$. So $\mathcal{B} \supseteq Orb(G_1^{(5)})$ and hence $\mathcal{B} = \binom{X}{k}$, which is excluded from the definition of a design to avoid triviality.

We summarize the above results as:

Theorem 3.6. *No graphical* 2-$(\binom{n}{2}, 5, \lambda)$ *design exists if* $n \geq 538$.

B. Upper Bound for Existence of Graphical 3-$(v, 5, \lambda)$ Designs

We prove in this section that no graphical 3-$(\binom{n}{2}, 5, \lambda)$ design exists if $n \geq 34$.

Let $(X, \mathcal{B})$ be a graphical 3-$(\binom{n}{2}, 5, \lambda)$ design, where $n \geq 34$. We may assume without loss of generality that $\mathcal{B} \supseteq Orb(G_7^{(5)})$, since otherwise we can consider the complement of the design. By considering the number of blocks in $Orb(G_7^{(5)})$ containing $G_1^{(3)}$, we see that

$$\lambda \geq \frac{1}{8}(n-3)^{\underline{4}}. \tag{7}$$

Lemma 3.7. $\mathcal{B} \supseteq \bigcup_{i\in\{20,21,22,26\}} Orb(G_i^{(5)})$.

Proof. Suppose that $\mathcal{B} \not\supseteq Orb(G_{20}^{(5)})$. Then by considering the number of blocks in $\mathcal{B}$ containing $G_2^{(3)}$, we have

$$\lambda \leq \binom{\binom{n}{2}-3}{2} - \frac{1}{8}(n-5)^{\underline{4}}.$$

The above inequality, together with inequality (7), implies

$$\frac{1}{8}(n-3)^{\underline{4}} \leq \binom{\binom{n}{2}-3}{2} - \frac{1}{8}(n-5)^{\underline{4}},$$

giving

$$(n-4)(n^3 - 38n^2 + 231n - 498) \leq 0,$$

which is impossible for $n \geq 32$.

To show that $\mathcal{B} \supseteq Orb(G_i^{(5)})$ for $i \in \{21, 22, 26\}$, mimic the proof above. □

It follows that $\mathcal{B} \supseteq \bigcup_{i\in\{7,20,21,22,26\}} Orb(G_i^{(5)})$. Let $\mathcal{A} = \binom{X}{5} \setminus \mathcal{B}$ and consider the 3-$(\binom{n}{2}, 5, \lambda')$ design $(X, \mathcal{A})$. By considering the number of blocks in $\mathcal{A}$ containing $G_5^{(3)}$, we see that

$$\lambda' \leq 12(n-6)^{\underline{2}} + 84(n-6) + 66. \tag{8}$$

Lemma 3.8. *$\mathcal{A} \not\supseteq Orb(G_i^{(5)})$ for $i \in \{2, 6, 11, 13, 16, 18, 24\}$.*

Proof. Suppose that $\mathcal{A} \supseteq Orb(G_2^{(5)})$. Then by considering the number of blocks in $Orb(G_2^{(5)})$ containing $G_1^{(3)}$, we have

$$\lambda' \geq \frac{3}{2}(n-3)^{\underline{3}}.$$

The above inequality, together with inequality (8), implies

$$\frac{3}{2}(n-3)^{\underline{3}} \leq 12(n-6)^{\underline{2}} + 84(n-6) + 66,$$

giving

$$n^3 - 20n^2 + 95n - 120 \leq 0,$$

which is impossible for $n \geq 14$.

To show that $A \not\supseteq Orb(G_i^{(5)})$ for $i \in \{6, 11, 13, 16, 18, 24\}$, mimic the proof above. □

It follows that $\mathcal{B} \supseteq \bigcup_{i\in\{2,6,7,11,13,16,18,20,21,22,24,26\}} Orb(G_i^{(5)})$. By considering the number of blocks in $\mathcal{A}$ containing $G_5^{(3)}$, we now have

$$\lambda' \leq 54. \tag{9}$$

Lemma 3.9. $\mathcal{A} \not\supseteq Orb(G_i^{(5)})$ *for* $i \in \{1, 3, 4, 5, 8, 9, 10, 12, 14, 15, 17, 19, 23, 25\}$.

Proof. Suppose that $\mathcal{A} \supseteq Orb(G_1^{(5)})$. Then by considering the number of blocks in $Orb(G_1^{(5)})$ containing $G_1^{(3)}$, we have

$$\lambda' \geq 3(n-3).$$

The above inequality, together with inequality (9), implies

$$3(n-3) \leq 54,$$

which is impossible for $n \geq 22$.

To show that $\mathcal{A} \not\supseteq Orb(G_i^{(5)})$ for $i \in \{3, 4, 5, 8, 9, 10, 12, 14, 15, 17, 19, 23, 25\}$, mimic the proof above. □

We can now conclude that $\mathcal{B} \supseteq \binom{X}{5}$, which is excluded from the definition of a design to avoid triviality. We summarize the above results as:

Theorem 3.10. *No graphical* 3-$(\binom{n}{2}, 5, \lambda)$ *design exists if* $n \geq 34$.

4. COMPUTATION FOR EXISTENCE

The symbolic computation approach of Chee [2] can, in theory, be used to find all graphical t-(v, k, λ) designs for given t and k, without the need to establish upper bounds for existence, such as in the previous section. However, in practice, the method becomes infeasible when k becomes large. Already for $k = 5$ we would have to solve up to 33 million systems of simultaneous Diophantine equations of degree up to six. Fortunately, using the upper bounds from the previous section, a straightforward exhaustive search suffices. In both of the cases $(t, k) = (2, 5)$ and $(t, k) = (3, 5)$, there are 26 possible orbits of 5-edge graphs, implying that we can easily enumerate all the 2^{26} =67,108,864 candidate designs, represented as $\{0, 1\}$-vectors u, and filter out those candidates that do not constitute a solution to the system

$$W_{t,5}(X|\mathcal{S}_n^{[2]})\mathsf{u} = \lambda(1, \ldots, 1)^\top, \quad \lambda \leq \frac{1}{2}\binom{\binom{n}{2} - t}{5 - t}.$$

In particular, this system needs to be considered in the two cases $t = 2$ and $t = 3$ for all $n \leq 537$ and $n \leq 39$, respectively. Both authors of this article independently carried out this computation with the following identical results.

A. Existence of Graphical 2-$(v, 5, \lambda)$ Designs

Our computations show that there are no graphical 2-$(\binom{n}{2}, 5, \lambda)$ designs for $40 \leq n \leq 537$. For $n \leq 39$, the number of inequivalent graphical 2-$(\binom{n}{2}, 5, \lambda)$ designs is fairly large, and for reasons of space, it is infeasible to give a complete listing within this article. A complete catalogue of the designs can be found on the first author's website at

⟨`http://www1.spms.ntu.edu.sg/~ymchee/graphical.htm`⟩.

We record this result as:

Theorem 4.1. *There are* 8,619 *elements in* Ψ(2, 5) *and there exist* 271,360 *inequivalent graphical* 2-($\binom{n}{2}$, 5, λ) *designs. No graphical* 2-($\binom{n}{2}$, 5, λ) *design exists if* $n \geq 40$.

B. Existence of Graphical 3-(v, 5, λ) Designs

Our computations show that there are no graphical 3-($\binom{n}{2}$, 5, λ) designs for $10 \leq n \leq 33$. For $n \leq 9$, a complete listing of all inequivalent graphical 3-($\binom{n}{2}$, 5, λ) designs found is presented below.

All elements of Ψ(3, 5) and inequivalent solutions		
(v, λ)	{0, 1}-vectors u^{T} giving inequivalent solutions	Number of inequivalent solutions
(15, 30)	10010100110000001000001000	1
(21, 3)	00000010000000100000000010	1
(21, 30)	00001100001001000000001100	1
(21, 33)	00001110001001100000001110	1
(21,39)	00010010100000010000000010 01000011010000101000000000 01000011010001100100000000	3
(21, 48)	10010000001100100010000010 10100000000100100110000010	2
(21, 69)	00011110101001010000001110 00101110100000011000001110 00101110100001010100001110 00101111000101010010001110 01001111011001101000001100	5
(21, 75)	01010011110000011000000000 01010011110001010100000000	2
(28, 30)	00000100000001100000001110	1
(28, 150)	00110101010100010110001000 00110101011000011010001000 11001010100111100100110110 11001010101011101000110110	4
(36, 180)	00101010011001000101001000	1
(36, 198)	11000000011001110010001110	1
(36, 258)	10101111000110111011000110 10110100011100100110100010 10110100101010101010100010	3

We record this result as:

Theorem 4.2. *There are* 13 *elements in* Ψ(3, 5) *and there exist* 26 *inequivalent graphical* 3-($\binom{n}{2}$, 5, λ) *designs. No graphical* 3-($\binom{n}{2}$, 5, λ) *design exists if* $n \geq 10$.

5. CONCLUSION

In this article, we determined completely the sets $\Psi(2, 5)$ and $\Psi(3, 5)$, and found more than 270,000 inequivalent graphical designs, and more than 8,000 new parameter sets for which there exists a graphical design.

We remark that our computation revealed two minor errors in [1]; in fact,

(i) there is only one graphical 2-(21, 5, λ) design for $\lambda = 52$ and $\lambda = 84$; and
(ii) there exist only two inequivalent (and hence at most two nonisomorphic) graphical 3-(21, 5, 75) designs.

A natural question is whether the techniques in this article could be developed further to determine $\Psi(t, k)$ for higher k, in particular for $k = 6$. The method for establishing upper bounds for existence is certainly applicable, but the main hurdle is the search for solutions to $W_{t,k}(X|\mathcal{S}_n^{[2]})\mathsf{u} = \lambda(1, \ldots, 1)^{\mathsf{T}}$. There are 68 nonisomorphic 6-edge graphs, so the naïve search space has size 2^{68}. More sophisticated search techniques must be employed in this case.

APPENDIX A

All Known Graphical *t*-Designs

TABLE I. Complete Knowledge of $\Psi(t, k)$

t	k	All elements of $\Psi(t, k)$					$\lvert\Psi(t, k)\rvert$
2	3	(10, 4)	(15, 1)	(28, 6)	(28, 10)	(55, 25)	5
2	4	(10, 2)	(10, 4)	(10, 8)	(10, 10)	(10, 12)	79
		(15, 6)	(15, 24)	(15, 30)	(15, 36)	(21, 6)	
		(21, 12)	(21, 18)	(21, 36)	(21, 42)	(21, 45)	
		(21, 48)	(21, 51)	(21, 54)	(21, 57)	(21, 60)	
		(21, 63)	(21, 66)	(21, 69)	(21, 72)	(21, 75)	
		(21, 78)	(21, 81)	(21, 84)	(28, 5)	(28, 55)	
		(28, 80)	(28, 85)	(28, 95)	(29, 110)	(28, 120)	
		(28, 125)	(28, 135)	(28, 150)	(36, 15)	(36, 90)	
		(36, 111)	(36, 120)	(36, 135)	(36, 165)	(36, 210)	
		(36, 231)	(36, 240)	(36, 255)	(36, 276)	(45, 63)	
		(45, 105)	(45, 252)	(45, 357)	(45, 378)	(45, 420)	
		(55, 168)	(55, 336)	(55, 504)	(78, 630)	(78, 1080)	
		(78, 1350)	(91, 836)	(91, 1430)	(91, 1496)	(105, 1320)	
		(105, 1326)	(105, 1650)	(105, 1656)	(105, 1782)	(105, 1788)	
		(105, 1980)	(105, 1986)	(105, 2112)	(105, 2118)	(105, 2442)	
		(105, 2448)	(153, 4935)	(153, 5025)	(253, 14535)		
3	4	(10, 1)					1
4	5	—					0
5	6	—					0

TABLE II. Partial Knowledge of $\Psi(t, k)$

t	k	Known elements of $\Psi(t, k)$					$\|\Psi(t, k)\| \geq$
2	5	(10, 16)	(10, 20)	(21, 7)	(21, 12)	(21, 19)	98
		(21, 22)	(21, 34)	(21, 35)	(21, 47)	(21, 50)	
		(21, 52)	(21, 55)	(21, 57)	(21, 60)	(21, 62)	
		(21, 64)	(21, 67)	(21, 69)	(21, 70)	(21, 72)	
		(21, 77)	(21, 79)	(21, 82)	(21, 84)	(21, 89)	
		(21, 94)	(21, 95)	(21, 100)	(21, 120)	(28, 60)	
		(28, 100)	(28, 140)	(28, 160)	(28, 200)	(28, 240)	
		(28, 260)	(28, 300)	(28, 340)	(28, 360)	(36, 60)	
		(36, 80)	(36, 140)	(36, 164)	(36, 180)	(36, 224)	
		(36, 240)	(36, 244)	(36, 480)	(36, 720)		
		$(15, \lambda) : 16 \leq \lambda \leq 142, \lambda \equiv 0, 2, 4,$ or 6 (mod 10), $\lambda \neq 20, 50$					
2	6	(21, 13)	(21, 30)	(21, 38)	(21, 45)	(21, 48)	78
		(21, 50)	(21, 51)	(21, 55)	(21, 58)	(21, 60)	
		(21, 61)	(21, 63)	(21, 68)	(21, 70)	(28, 25)	
		(28, 40)	(28, 50)	(28, 65)	(28, 70)	(28, 80)	
		(28, 90)	(28, 100)	(36, 20)	(36, 45)	(36, 120)	
		(36, 240)	(36, 540)	(36, 720)	(36, 1080)	(36, 2160)	
		(36, 4320)					
		$(15, \lambda) : 10 \leq \lambda \leq 355, \lambda \equiv 0$ or 10 (mod 15)					
2	7	(15, 3)	(15, 24)	(15, 27)	(15, 30)	(15, 33)	(224)
		(15, 36)	(15, 39)	(21, 42)	(21, 63)	(21, 78)	
		(21, 84)	(21, 105)	(28, 16)	(28, 140)	(28, 156)	
		(28, 182)	(28, 198)	(36, 210)	(36, 246)	(36, 336)	
		(36, 372)	(36, 420)	(36, 456)	(36, 462)	(36, 546)	
		$(15, \lambda) : 48 \leq \lambda \leq 642, \lambda \equiv 0$ (mod 3)					
2	8	(21, 84)	(21, 168)	(21, 336)	(21, 672)	(28, 70)	6
		(28, 210)					
2	9	(21, 12)	(21, 54)	(21, 72)	(21, 108)	(21, 216)	15
		(21, 432)	(21, 864)	(28, 40)	(28, 160)	(28, 320)	
		(28, 480)	(28, 640)	(28, 960)	(28, 1920)	(28, 3840)	
3	5	(15, 30)	(21, 3)	(21, 30)	(21, 33)	(21, 39)	12
		(21, 48)	(21, 69)	(21, 75)	(28, 30)	(28, 150)	
		(36, 180)	(36, 270)				
3	6	(15, 100)	(21, 68)	(21, 100)	(21, 108)	(21, 128)	22
		(21, 136)	(21, 140)	(21, 148)	(21, 156)	(21, 160)	
		(21, 168)	(21, 176)	(21, 180)	(21, 188)	(21, 196)	
		(21, 200)	(28, 80)	(28, 120)	(28, 180)	(28, 220)	
		(28, 240)	(28, 260)				
3	7	(15, 60)	(15, 75)	(15, 90)	(15, 135)	(15, 150)	18
		(15, 165)	(15, 180)	(15, 225)	(15, 240)	(21, 105)	
		(21, 120)	(21, 210)	(21, 225)	(21, 315)	(28, 210)	
		(28, 225)	(28, 240)	(28, 275)			

(*Continued*)

TABLE II. **(*Continued*)**

t	k	Known elements of $\Psi(t, k)$					$\|\Psi(t, k)\| \geq$
3	8	(21, 168)	(21, 252)	(21, 336)	(21, 420)	(28, 168)	7
		(28, 378)	(28, 672)				
3	9	(28, 280)					1
4	6	(28, 132)					1
4	7	(15, 60)					1
5	7	(28, 93)	(36, 165)				2
5	8	(28, 756)	(28, 791)	(28, 840)	(28, 875)		4

APPENDIX B

Orbit Representatives

A list of orbit representatives for t-edge graphs, for $t = 2$, $t = 3$, and $t = 5$, is given below. Note that isolated vertices are not shown in our drawings. The orbit representative indexing row i of $W_{t,5}(X|\mathcal{S}_n^{[2]})$ is the graph $G_i^{(t)}$, $t \in \{2, 3\}$, and the orbit representative indexing column j of $W_{t,5}(X|\mathcal{S}_n^{[2]})$ is the graph $G_j^{(5)}$.

TABLE III. **Orbit Representatives of 2-Edge Graphs**

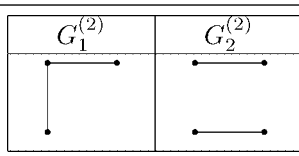


TABLE IV. **Orbit Representatives of 3-Edge Graphs**

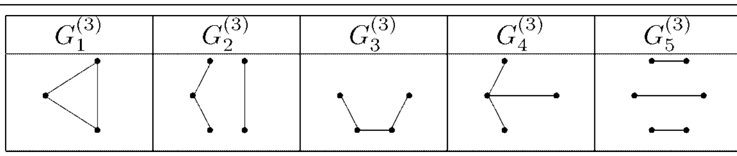
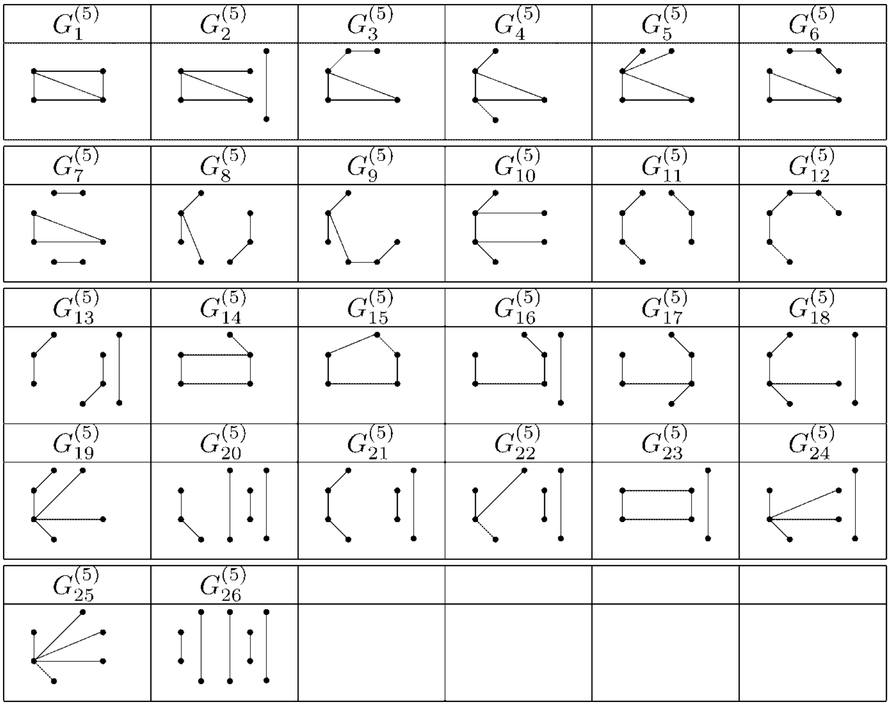


TABLE V. Orbit Representatives of 5-Edge Graphs



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