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DECOMPOSITIONS OF EDGE-COLORED DIGRAPHS: A NEW TECHNIQUE IN THE CONSTRUCTION OF CONSTANT-WEIGHT CODES AND RELATED FAMILIES*

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Abstract. We demonstrate that certain Johnson-type bounds are asymptotically exact for a variety of classes of codes, namely, constant-composition codes, nonbinary constant-weight codes, group divisible codes, and multiply constant-weight codes. We achieve this via an application of the theory of decomposition of edge-colored digraphs.

Key words. edge-colored digraphs, constant-weight codes, constant-composition codes, multiply constant-weight codes

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1. Introduction.

Binary constant-weight codes form an important class of codes where every codeword has the same weight. A fundamental problem in this study is to determine $A(n, d, w)$, the maximum number of codewords in a constant-weight code of length n with Hamming distance d and constant-weight w [24]. In recent years, binary constant-weight codes have been generalized to constant-composition codes and nonbinary constant-weight codes and have found applications in coding for bandwidth-efficient channels [14], powerline communications [11, 8], and frequency hopping [12]. More recently, the class of multiply constant-weight codes was introduced in applications for physically unclonable functions [1] and simultaneous energy and information transfer [29]. In these studies, quantities analogous to $A(n, d, w)$ have been defined, and these quantities are objects of interest.

Constant-weight codes have deep connections with many combinatorial objects in design theory, such as *Steiner systems* and *packing designs* [24, 13]. In the 1970s, Wilson [32, 33, 31, 34] demonstrated that the elementary necessary conditions for the existence of balanced incomplete block designs, a special case of Steiner systems, are asymptotically sufficient as well. His work, therefore, determined the size of an optimal binary constant-weight code of weight w and distance $2w - 2$, provided that the length n is sufficiently large and satisfies certain congruence conditions. More

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recently, Keevash [21] demonstrated the existence of general Steiner systems. His result implies that the Johnson bound for binary constant-weight codes is exact, whenever the distance is fixed, and the code length is sufficiently large and satisfies certain congruence conditions.

Since Wilson's seminal work, there have been interesting developments in the area of combinatorial designs. Wilson's ideas have been generalized into Lamken and Wilson's theory of *decomposition of edge-colored digraphs* [23], and the theory has been used extensively in establishing the asymptotic existence of many combinatorial designs. Theoretical developments also extend Lamken and Wilson's results to other types of decompositions. Of particular interest is the class of *superpure* decompositions [20].

Therefore, a natural question is whether these advanced techniques in the decomposition of edge-colored digraphs are relevant in constructing optimal codes in these generalizations. In this paper, we answer this question in the affirmative for certain distances. Specifically, we construct families of optimal codes by establishing the connection between these codes and the decompositions of edge-colored digraphs.

The rest of this paper is organized as follows. In section 2, we define relevant terminology in detail, present the known results of these codes, and summarize our contributions. In sections 3–6, we establish the connections between these three classes of codes and the decomposition of edge-colored digraphs, and prove that some bounds—in particular, the Johnson-type bounds—can be reached for sufficiently large code lengths. Related open problems are discussed in section 7.

2. Preliminary.

The ring $\mathbb{Z}/q\mathbb{Z}$ is denoted by $\mathbb{Z}_q$. For positive integer n, the set $\{1, 2, \ldots, n\}$ is denoted by $[n]$.

Let $\mathbb{Z}_q^X$ denote the set of vectors whose elements belong to $\mathbb{Z}_q$ and are indexed by X. A *q-ary code of length n* is then a set $\mathcal{C} \subseteq \mathbb{Z}_q^X$ with $|X| = n$. The *support* of a vector $\mathsf{u} \in \mathbb{Z}_q^X$, denoted $\mathrm{supp}(\mathsf{u})$, is the set $\{x \in X : \mathsf{u}_x \neq 0\}$. The *Hamming weight* of $\mathsf{u} \in \mathbb{Z}_q^X$ is defined as $\|\,\mathsf{u}\,\| = |\mathrm{supp}(\mathsf{u})|$. The distance induced by this norm is called the *Hamming distance*, so that $d(\mathsf{u}, \mathsf{v}) = \|\,\mathsf{u} - \mathsf{v}\,\|$, for $\mathsf{u}, \mathsf{v} \in \mathbb{Z}_q^X$. A code $\mathcal{C}$ is said to have *distance d* if $d(\mathsf{u}, \mathsf{v}) \geq d$ for all distinct $\mathsf{u}, \mathsf{v} \in \mathcal{C}$. The *composition* of a vector $\mathsf{u} \in \mathbb{Z}_q^X$ is the tuple $\overline{w} = [w_1, \ldots, w_{q-1}]$, where $w_i = |\{x \in X : \mathsf{u}_x = i\}|$, where $i \in \mathbb{Z}_q \setminus \{0\}$. Unless mentioned otherwise, we always assume $q \geq 3$ and $w_1 \geq w_2 \geq \cdots \geq w_{q-1}$.

2.1. Constant-weight codes and constant-composition codes.

A code $\mathcal{C}$ is said to have *constant-weight* w if every codeword in $\mathcal{C}$ has weight w, and to have *constant-composition* $\overline{w}$ if every codeword in $\mathcal{C}$ has composition $\overline{w}$. We refer to a q-ary code of length n, distance d, and constant-weight w as a $\mathrm{CWC}(n, d, w)_q$. If in addition, the code has constant-composition $\overline{w}$, then it is referred to as a $\mathrm{CCC}(n, d, \overline{w})_q$. The maximum size of a $\mathrm{CWC}(n, d, w)_q$ is denoted $A_q(n, d, w)$, while the maximum size of a $\mathrm{CCC}(n, d, \overline{w})_q$ is denoted $A_q(n, d, \overline{w})$. Any $\mathrm{CWC}(n, d, w)_q$ or $\mathrm{CCC}(n, d, \overline{w})_q$ attaining the maximum size is called *optimal*.

Consider a composition $\overline{w}$ and let $w = \sum_{i=1}^{q-1} w_i$. Then a $\mathrm{CCC}(n, d, \overline{w})_q$ is also a $\mathrm{CWC}(n, d, w)_q$.

The following Johnson-type bounds for q-ary constant-weight codes and constant-composition codes have been derived.

LEMMA 2.1 (Svanström [26]; Svanström, Östergård, and Bogdanova [28]).

$$A_q(n,d,\overline{w}) \le \left\lfloor \frac{n}{w_1} A_q(n-1,d,[w_1-1,\dots,w_{q-1}]) \right\rfloor,$$
$$A_q(n,d,w) \le \left\lfloor \frac{(q-1)n}{w} A_q(n-1,d,w-1) \right\rfloor.$$

Applying the facts that $A_q(n,2w,\overline{w}) = \lfloor n/w \rfloor$ and $A_q(n,2w,w) = \lfloor n/w \rfloor$ (see Fu, Vinck, and Shen [16] and Chee, Ge, and Ling [6]) to Lemma 2.1, we have the following upper bounds:

$$A_q(n,2w-2,\overline{w}) \le \left\lfloor \frac{n}{w_1} \left\lfloor \frac{n-1}{w-1} \right\rfloor \right\rfloor, \tag{1}$$

$$A_q(n,2w-3,\overline{w}) \le \begin{cases} \left\lfloor \frac{n}{w_1} \left\lfloor \frac{n-1}{w_1-1} \right\rfloor \right\rfloor & \text{if } w_1 > w_2, \\ \left\lfloor \frac{n}{w_1} \left\lfloor \frac{n-1}{w_1} \right\rfloor \right\rfloor & \text{otherwise,} \end{cases} \tag{2}$$

$$A_q(n,2w-2,w) \le \left\lfloor \frac{(q-1)n}{w} \left\lfloor \frac{n-1}{w-1} \right\rfloor \right\rfloor, \tag{3}$$

$$A_q(n,2w-3,w) \le \left\lfloor \frac{(q-1)n}{w} \left\lfloor \frac{(q-1)(n-1)}{w-1} \right\rfloor \right\rfloor. \tag{4}$$

We list some previous asymptotic or exact results.

(i) Results for $A_q(n,d,\overline{w})$ are known
  (a) for all $\overline{w}$ and $d = 2w-1$ [4, 10];
  (b) for all d where $w \le 3$ [27, 4];
  (c) for $(q,d,w) = (3,5,4)$ [17].

(ii) Results for $A_q(n,d,w)$ are known
  (a) for all w and $d = 2w-1$ [9, 4];
  (b) for all q and $(d,w) \in \{(3,2),(4,3),(5,3)\}$ [9, 3, 7];
  (c) for $(q,d,w) \in \{(3,5,4),(3,6,4),(4,5,4)\}$ [35, 37, 36].

Our contributions. In this paper, we show that the inequalities (1)–(4) are exact provided that n is sufficiently large and n satisfies certain congruence conditions. In other words, for any fixed composition $\overline{w}$ or fixed weight w, we determine $A_q(n,d,\overline{w})$ and $A_q(n,d,w)$ for $d \in \{2w-2, 2w-3\}$ when n is sufficiently large and satisfies certain congruence conditions.

2.2. Multiply constant-weight codes. Consider a binary code $\mathcal{C} \subseteq \mathbb{Z}_2^{[m]\times[n]}$ of constant-weight mw and distance d. The code $\mathcal{C}$ is said to be of *multiply constant-weight* w if for $\mathsf{u} \in \mathcal{C}$, $i \in [m]$, the subword $(\mathsf{u}_{i,j})_{j\in[n]}$ is of constant-weight w. Denote such a code by MCWC(m,n,d,w). Similarly, the maximum size of an MCWC(m,n,d,w) is given by $M(m,n,d,w)$, and a multiply constant-weight code attaining the maximum size is said to be *optimal.* The following bounds can be derived from [1].

LEMMA 2.2 (Chee et al. [1]).

$$M(m,n,2mw-2,w) \le \left\lfloor \frac{n}{w} \left\lfloor \frac{n}{w} \right\rfloor \right\rfloor, \tag{5}$$

$$M(m,n,d,w) \le \left\lfloor \frac{d/2}{d/2 + mw^2/n - mw} \right\rfloor. \tag{6}$$

Our contributions. Again, we verify that the upper bound (5) in Lemma 2.2 is exact provided n is sufficiently large for some cases, and the upper bound (6) is also exact for a large range of parameters. In particular, we demonstrate the following:

(i) When distance $d = 2mw-2$ for any fixed m and w, we determine $M(m,n,d,w)$ provided that n is sufficiently large and satisfies certain congruence conditions.

(ii) When distance $d = 2(mw-\lambda)$ for any fixed w and λ, we determine $M(m,n,d,w)$ provided that n is some function of m and m is sufficiently large.

Next, we describe the main tool in our constructions of optimal codes.

2.3. Decomposition of edge-colored complete digraphs. Denote the set of all ordered pairs of a finite set X with distinct components by $\overline{\binom{X}{2}}$. An *edge-colored digraph* is a triple $G=(V,C,E)$, where V is a finite set of *vertices*, C is a finite set of *colors,* and E is a subset of $\overline{\binom{V}{2}} \times C$. Members of E are called *edges*. The *complete edge-colored digraph* on n vertices with r colors, denoted by $K_n^{(r)}$, is the edge-colored digraph (V,C,E), where $|V|=n$, $|C|=r$, and $E=\overline{\binom{V}{2}}\times C$.

A family $\mathcal{F}$ of edge-colored subgraphs of an edge-colored digraph K is a *decomposition* of K if every edge of K belongs to exactly one member of $\mathcal{F}$. Given a family of edge-colored digraphs $\mathcal{G}$, a decomposition $\mathcal{F}$ of K is a *$\mathcal{G}$-decomposition of* K if each edge-colored digraph in $\mathcal{F}$ is isomorphic to some $G \in \mathcal{G}$. Furthermore, a $\mathcal{G}$-decomposition of K is said to be *superpure* if any two distinct edge-colored subgraphs in $\mathcal{F}$ share at most two vertices.

Lamken and Wilson [23] exhibited the asymptotic existence of decompositions of $K_n^{(r)}$ for a fixed family of digraphs. Hartmann [20] later extended their results to superpure decompositions. To state the theorems, we require more concepts.

Consider an edge-colored digraph $G=(V,C,E)$ with $|C|=r$. Let $((u,v),c)\in E$ denote a directed edge from u to v, colored by c. For any vertex u and color c, define the *indegree* and *outdegree* of u with respect to c as the number of directed edges of color c entering and leaving u, respectively. Then for vertex u, we define the *degree vector* of u in G, denoted by $\tau(u,G)$, as the vector of length $2r$, $\tau(u,G) \triangleq (\mathrm{in}_1(u,G), \mathrm{out}_1(u,G), \dots, \mathrm{in}_r(u,G), \mathrm{out}_r(u,G))$. Define $\alpha(\mathcal{G})$ as the greatest common divisor of the integers t such that the $2r$-vector $(t,t,\dots,t)$ is a nonnegative integral linear combination of the degree vectors $\tau(u,G)$ as u ranges over all vertices of all digraphs $G\in\mathcal{G}$.

For each $G=(V,C,E)\in\mathcal{G}$, let $\mu(G)$ be the *edge vector* of length r given by $\mu(G)\triangleq(m_1(G),m_2(G),\dots,m_r(G))$, where $m_i(G)$ is the number of edges with color i in G. We denote by $\beta(\mathcal{G})$ the greatest common divisor of the integers m such that the r-vector $(m,m,\dots,m)$ is a nonnegative integral linear combination of the vectors $\mu(G)$, $G\in\mathcal{G}$. Then $\mathcal{G}$ is said to be *admissible* if $(1,1,\dots,1)$ can be expressed as a positive rational combination of the vectors $\mu(G)$, $G\in\mathcal{G}$.

Below is the main theorem we allude to in our proofs.

THEOREM 2.3 (Lamken and Wilson [23]; Hartmann [20]). *Let $\mathcal{G}$ be an admissible family of edge-colored digraphs with r colors. Then there exists a constant n_0 (resp., n_1) such that a (resp., superpure) $\mathcal{G}$-decomposition of $K_n^{(r)}$ exists for every $n\ge n_0$ (resp., $n\ge n_1$) satisfying $n(n-1)\equiv 0 \pmod{\beta(\mathcal{G})}$ and $n-1\equiv 0 \pmod{\alpha(\mathcal{G})}$.*

We apply Theorem 2.3 to construct codes that meet the upper bounds (1)–(6). To do so, we have two main steps.

(A) Define a family $\mathcal{G}$ of edge-colored digraphs on a set of r colors. We then show that a $\mathcal{G}$-decomposition of $K_n^{(r)}$ results in a code of length n satisfying certain weight and distance properties.

(B) Compute $\alpha(\mathcal{G})$ and $\beta(\mathcal{G})$ and hence determine the congruence conditions that n needs to satisfy.

In certain cases, it may not be easy to compute the exact values of $\alpha(\mathcal{G})$ and $\beta(\mathcal{G})$. In these cases, we then demonstrate that $\alpha(\mathcal{G})$ and $\beta(\mathcal{G})$ divide some α' and β', respectively. Observe that whenever α' and β' divide $n-1$ and $n(n-1)$, respectively, we have that $\alpha(\mathcal{G})$ and $\beta(\mathcal{G})$ also divide $n-1$ and $n(n-1)$, respectively. Hence, we may obtain a weaker version of Theorem 2.3.

COROLLARY 2.4. *Let $\mathcal{G}$ be an admissible family of edge-colored digraphs with r colors. Suppose that $\alpha(\mathcal{G}) \mid \alpha'$ and $\beta(\mathcal{G}) \mid \beta'$ for some α' and β'. Then there exists a constant n_0 (resp., n_1) such that a (resp., superpure) $\mathcal{G}$-decomposition of $K_n^{(r)}$ exists for every $n \geq n_0$ (resp., $n \geq n_1$) satisfying $n(n-1) \equiv 0 \pmod{\beta'}$ and $n-1 \equiv 0 \pmod{\alpha'}$.*

3. Asymptotic exactness of Johnson-type bounds for codes with constant composition.

Fix $\overline{w} = [w_1, w_2, \ldots, w_{q-1}]$ with $w_1 \geq \cdots \geq w_{q-1} > 0$, and let $w = \sum_{i=1}^{q-1} w_i$. In this section, we construct infinite families of optimal $\mathrm{CCC}(n, d, \overline{w})_q$ for $d = 2w-2$ or $d = 2w-3$ and establish the asymptotic exactness of (1) and (2).

3.1. When distance $d = 2w - 2$.

Consider the following characterization of codes of constant-weight w with distance $2w-2$. Note that since the constant-composition codes are constant-weight codes, the lemma is applicable for both classes of codes.

LEMMA 3.1. *The following are necessary and sufficient for a code $\mathcal{C}$ of constant-weight w to have distance $2w-2$:*

(C1) *For $i \in [q-1]$, the ordered pairs in the set $\{(x,y) : \mathsf{u}_x = i, y \in \mathrm{supp}(\mathsf{u}) \setminus \{x\}, \text{ for any } \mathsf{u} \in \mathcal{C}\}$ are distinct; and*

(C2) *for any $\mathsf{u}, \mathsf{v} \in \mathcal{C}$, $|\mathrm{supp}(\mathsf{u}) \cap \mathrm{supp}(\mathsf{v})| \leq 2$.*

Proof. Suppose that a code $\mathcal{C}$ of constant-weight w has distance $2w-2$. Now we show that (C1) and (C2) hold for $\mathcal{C}$. For (C1), assume that for some $i \in [q-1]$, an ordered pair (x,y) appears twice in the set $\{(x,y) : \mathsf{u}_x = i, y \in \mathrm{supp}(\mathsf{u}) \setminus \{x\}, \text{ for any } \mathsf{u} \in \mathcal{C}\}$. That means that there exist some distinct $\mathsf{u}, \mathsf{v} \in \mathcal{C}$, such that $\mathsf{u}_x = \mathsf{v}_x = i$, $\mathsf{u}_y \neq 0$, and $\mathsf{v}_y \neq 0$. Hence, $d(\mathsf{u}, \mathsf{v}) \leq 2(w-2) + 1 < 2w-2$, which contradicts the distance. For (C2), assume there exist distinct $\mathsf{u}, \mathsf{v} \in \mathcal{C}$ such that $|\mathrm{supp}(\mathsf{u}) \cap \mathrm{supp}(\mathsf{v})| \geq 3$; then $d(\mathsf{u}, \mathsf{v}) \leq 2w - |\mathrm{supp}(\mathsf{u}) \cap \mathrm{supp}(\mathsf{v})| \leq 2w-3 < 2w-2$, which also causes a contradiction. Therefore, the conditions (C1) and (C2) hold for the code $\mathcal{C}$.

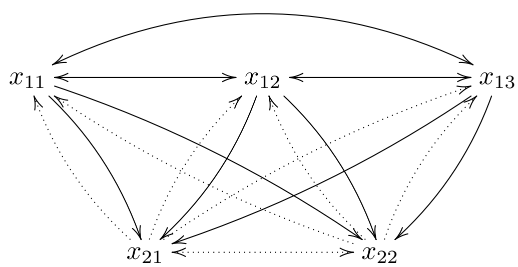
For the converse, suppose conditions (C1) and (C2) hold for a code $\mathcal{C}$ of constant-weight w. Assume there exist distinct $\mathsf{u}, \mathsf{v} \in \mathcal{C}$ such that $d(\mathsf{u}, \mathsf{v}) \leq 2w-3$; then we have either $|\mathrm{supp}(\mathsf{u}) \cap \mathrm{supp}(\mathsf{v})| = 2$, $\mathsf{u}_x = \mathsf{v}_x \neq 0$, $\mathsf{u}_y \neq 0$, and $\mathsf{v}_y \neq 0$ for some $x \neq y$, or $|\mathrm{supp}(\mathsf{u}) \cap \mathrm{supp}(\mathsf{v})| \geq 3$, which contradict either condition (C1) or (C2). Therefore, the distance of the code $\mathcal{C}$ is at least $2w-2$. ∎

Definition of the family $\mathcal{G}(\overline{w})$. For any fixed $\overline{w}$, we define an edge-colored digraph $G(\overline{w}) = (V(\overline{w}), C(\overline{w}), E(\overline{w}))$, where

$$V(\overline{w}) \triangleq \{x_{ij} : i \in [q-1], j \in [w_i]\};$$
$$C(\overline{w}) \triangleq [q-1];$$
$$E(\overline{w}) \triangleq \left\{((x_{ij}, x_{ij'}), i) : i \in [q-1], (j, j') \in \overline{\binom{[w_i]}{2}}\right\} \tag{7}$$
$$\bigcup \left\{((x_{ij}, x_{i'j'}), i) : (i, i') \in \overline{\binom{[q-1]}{2}}, j \in [w_i], j' \in [w_{i'}]\right\}.$$

Let s be the largest integer such that $w_1 = w_2 = \cdots = w_s$. For each $s+1 \le i \le q-1$, let G_i be an edge-colored digraph with two vertices y_i, z_i and one directed edge with color i from y_i to z_i. Then $\mathcal{G}(\overline{w}) = \{G(\overline{w})\} \cup \{G_i : s+1 \le i \le q-1\}$.

Example 3.2. Let $q = 3$ and $\overline{w} = [w_1, w_2] = [3, 2]$. The edge-colored digraph $G(\overline{w})$ is given below, where the solid lines denote directed edges with color 1, the dotted lines denote the directed edges with color 2, and " $\longleftrightarrow$ " (" $\leftarrow\cdots\rightarrow$ ") denotes two directed edges with the same color in each direction.



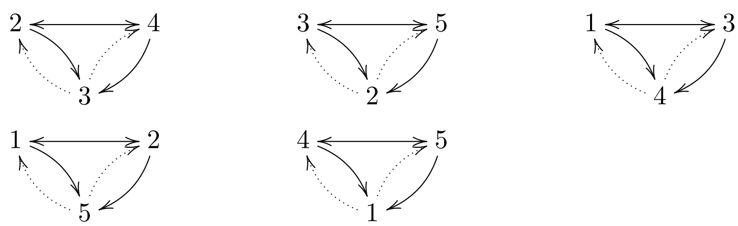
Then G_2 is the digraph $y_2 \dashrightarrow z_2$, and the family of digraphs is given by $\mathcal{G}(\overline{w}) = \{G(\overline{w}), G_2\}$.

Construction of an optimal $\mathbf{CCC}(n, 2w-2, \overline{w})_q$. Suppose a superpure $\mathcal{G}(\overline{w})$-decomposition of $K_n^{(q-1)}$ exists. For each F isomorphic to $G(\overline{w})$, there is a unique partition of the vertex set $V(F) = \bigcup_{i=1}^{q-1} S_i$ so that all outgoing edges from x in F have color i if and only if $x \in S_i$. Then construct one codeword u with support $V(F)$ such that $\mathsf{u}_x = i$ for $x \in S_i$. Hence, u has composition $\overline{w}$. Let $\mathcal{C}$ be the code consisting of all the codewords u constructed in this way. Now, we check that $\mathcal{C}$ is the desired code.[1]

Since we have a $\mathcal{G}(\overline{w})$-decomposition of $K_n^{(q-1)}$, for any $i \in [q-1]$, the pairs $(x, y) \in \overline{\binom{[n]}{2}}$ in the edges $((x, y), i)$ with color i are distinct; that is, the pairs in $\{(x, y) : \mathsf{u}_x = i, y \in \mathrm{supp}(\mathsf{u}) \setminus \{x\},$ for any $\mathsf{u} \in \mathcal{C}\}$ are all distinct by the definition of $G(\overline{w})$, and then (C1) of Lemma 3.1 is satisfied. Since the decomposition is superpure, any distinct F, F' isomorphic to $G(\overline{w})$ in the decomposition share at most two vertices, that is, $|V(F) \cap V(F')| \le 2$. Let $\mathsf{u}, \mathsf{v} \in \mathcal{C}$ be the codewords obtained from F and F'; then we have $|\mathrm{supp}(\mathsf{u}) \cap \mathrm{supp}(\mathsf{v})| \le 2$, and (C2) is also met. Furthermore, since each $G(\overline{w})$ uses $w_1(w-1)$ edges of color 1, and the total number of edges of color 1 in $K_n^{(q-1)}$ is $n(n-1)$, we obtain that $\mathcal{C}$ is a $\mathrm{CCC}(n, 2w-2, \overline{w})_q$ code of size $n(n-1)/(w_1(w-1))$ by Lemma 3.1. The code $\mathcal{C}$ meets the upper bound in (1) and is therefore optimal.

[1] While we only use the $G(\overline{w})$'s in the construction of the codewords, the G_i's are needed to ensure that *every edge* of $K_n^{(q-1)}$ appears in the $\mathcal{G}(\overline{w})$-decomposition.

Example 3.3. Let $\overline{w} = [2, 1]$. Consider the following superpure $\mathcal{G}(\overline{w})$-decomposition of $K_5^{(2)}$:



$1 \dashrightarrow 2$ $\quad$ $2 \dashrightarrow 1$ $\quad$ $3 \dashrightarrow 1$ $\quad$ $4 \dashrightarrow 2$ $\quad$ $5 \dashrightarrow 3$

$1 \dashrightarrow 3$ $\quad$ $2 \dashrightarrow 4$ $\quad$ $3 \dashrightarrow 5$ $\quad$ $4 \dashrightarrow 5$ $\quad$ $5 \dashrightarrow 4$

The corresponding code is then given by $\{(0, 1, 2, 1, 0), (0, 2, 1, 0, 1), (1, 0, 1, 2, 0), (1, 1, 0, 0, 2), (2, 0, 0, 1, 1)\}$, which is indeed an optimal $\mathrm{CCC}(5, 4, [2, 1])_3$.

Computation of $\boldsymbol{\alpha(\mathcal{G}(\overline{w}))}$ and $\boldsymbol{\beta(\mathcal{G}(\overline{w}))}$. First, consider the digraph $G(\overline{w})$. Observe that for $i, j \in [q-1], k \in [w_i]$ we have $\mathrm{in}_i(x_{ik}, G(\overline{w})) = w_i - 1$, $\mathrm{out}_i(x_{ik}, G(\overline{w})) = w - 1$, $\mathrm{in}_j(x_{ik}, G(\overline{w})) = w_j$, and $\mathrm{out}_j(x_{ik}, G(\overline{w})) = 0$ for $j \neq i$. Consider G_i for $s + 1 \leq i \leq q - 1$. Then $\mathrm{in}_i(z_i, G_i) = \mathrm{out}_i(y_i, G_i) = 1$ and all other indegrees and outdegrees are zero.

Let $a = \gcd(w_1, w)$. Pick $t = \lfloor w/w_1 \rfloor$ so that $0 \leq w - tw_1 < w_1$. Observe also that $t \geq s$. Consider the vector

$$\upsilon = \frac{w - tw_1}{a}\tau(x_{(t+1)1}, G(\overline{w})) + \frac{w_1}{a}\sum_{i=1}^{t}\tau(x_{i1}, G(\overline{w})).$$

For $j \in [q - 1]$, let $\mathrm{in}_j(\upsilon)$ and $\mathrm{out}_j(\upsilon)$ denote the coordinates in υ corresponding to the summation of the indegrees or outdegrees with respect to color j. Then we have

$$\mathrm{in}_j(\upsilon) = \begin{cases} \frac{ww_j - w_1}{a} = \frac{w_1(w-1)}{a} & \text{for } j \in [s], \\ \frac{ww_j - w_1}{a} < \frac{w_1(w-1)}{a} & \text{for } s + 1 \leq j \leq t, \\ \frac{ww_{t+1} - w + tw_1}{a} \leq \frac{ww_{t+1}}{a} & \text{for } j = t + 1, \\ \frac{ww_j}{a} & \text{otherwise,} \end{cases}$$

$$\mathrm{out}_j(\upsilon) = \begin{cases} \frac{w_1(w-1)}{a} & \text{for } j \in [t], \\ \frac{(w - tw_1)(w-1)}{a} < \frac{w_1(w-1)}{a} & \text{for } j = t + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the first $2s$ coordinates of υ are $w_1(w - 1)/a$ and all other coordinates have values at most $w_1(w - 1)/a$. Adding to υ a suitable nonnegative integral combination of $\tau(y_i, G_i)$'s and $\tau(z_i, G_i)$'s, we get $w_1(w - 1)/a \cdot (1, 1, \ldots, 1)$. Hence, we conclude that $\alpha(\mathcal{G}(\overline{w}))$ divides $w_1(w - 1)/a$.

Next, consider the edge vector $\mu(G)$ with $G \in \mathcal{G}(\overline{w})$. For $G(\overline{w})$, $m_i(G(\overline{w})) = w_i(w - 1)$ for $i \in [q - 1]$, while for G_i with $s + 1 \leq i \leq q - 1$, $m_i(G_i) = 1$ and $m_j(G_i) = 0$ for $j \neq i$. Consider the vector

$$\upsilon' = \mu(G(\overline{w})) + \sum_{i=s+1}^{q-1}(w_1(w - 1) - m_i(G(\overline{w})))\mu(G_i).$$

Since each coordinate of v' is $w_1(w-1)$, we have that $\beta(\mathcal{G}(\overline{w}))$ divides $w_1(w-1)$. It is obvious that $\frac{1}{w_1(w-1)}v'$ is the all one vector of positive rational combination of $\{\mu(G) : G \in \mathcal{G}(\overline{w})\}$. Hence, we have that $\mathcal{G}(\overline{w})$ is admissible.

Finally, applying Corollary 2.4, we obtain our first asymptotic result.

Proposition 3.4. *Fix $\overline{w}$ and let $w = \sum_{i=1}^{q-1} w_i$. There exists an integer n_0 such that*

$$A_q(n, 2w-2, \overline{w}) = \frac{n(n-1)}{w_1(w-1)}$$

for all $n \geq n_0$ satisfying $n(n-1) \equiv 0 \pmod{w_1(w-1)}$ and $n-1 \equiv 0 \pmod{w_1(w-1)/a}$, where $a = \gcd(w_1, w)$.

3.2. When distance $d = 2w-3$. We have the following analogous characterization of codes of constant-weight w with distance $2w-3$.

Lemma 3.5. *The following are sufficient for a code $\mathcal{C}$ of weight w to have distance $2w-3$:*

(C3) *For $i, j \in [q-1]$, the ordered pairs in the set $\{(x, y) : \mathsf{u}_x = i, \mathsf{u}_y = j, x \neq y, \text{ for any } \mathsf{u} \in \mathcal{C}\}$ are distinct; and*

(C4) *for any $\mathsf{u}, \mathsf{v} \in \mathcal{C}$, $|\mathrm{supp}(\mathsf{u}) \cap \mathrm{supp}(\mathsf{v})| \leq 2$.*

Proof. Suppose conditions (C3) and (C4) hold for a code $\mathcal{C}$ of weight w. Assume there exist distinct $\mathsf{u}, \mathsf{v} \in \mathcal{C}$ such that $d(\mathsf{u}, \mathsf{v}) \leq 2w-4$; then we have either $|\mathrm{supp}(\mathsf{u}) \cap \mathrm{supp}(\mathsf{v})| = 2$, $\mathsf{u}_x = \mathsf{v}_x \neq 0$ and $\mathsf{u}_y = \mathsf{v}_y \neq 0$ for some $x \neq y$, or $|\mathrm{supp}(\mathsf{u}) \cap \mathrm{supp}(\mathsf{v})| \geq 3$, which contradict either condition (C1) or (C2). Therefore, the distance of the code $\mathcal{C}$ is at least $2w-3$. ∎

Definition of the family $\mathcal{G}^*(\overline{w})$. For any fixed $\overline{w}$, we define an edge-colored digraph $G^*(\overline{w}) = (V^*(\overline{w}), C^*(\overline{w}), E^*(\overline{w}))$, where

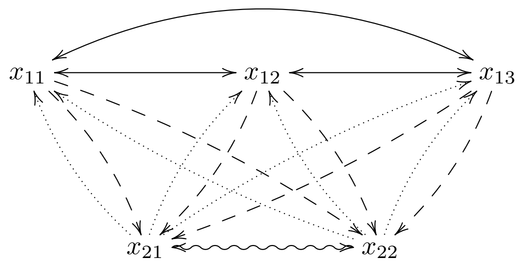
$$V^*(\overline{w}) \triangleq \{x_{ij} : i \in [q-1], j \in [w_i]\};$$
$$C^*(\overline{w}) \triangleq [q-1] \times [q-1];$$
$$E^*(\overline{w}) \triangleq \left\{ ((x_{ij}, x_{ij'}), (i,i)) : i \in [q-1], (j, j') \in \overline{\binom{[w_i]}{2}} \right\}$$
$$\bigcup \left\{ ((x_{ij}, x_{i'j'}), (i,i')) : (i, i') \in \overline{\binom{[q-1]}{2}}, j \in [w_i], j' \in [w_{i'}] \right\}. \tag{8}$$

For $i, j \in [q-1]$, let G^*_{ij} be a digraph with vertices y_{ij}, z_{ij} and one directed edge with color (i, j) from y_{ij} to z_{ij}. To define $\mathcal{G}^*(\overline{w})$, we have the following two cases depending on whether $w_1 = w_2$:

(i) When $w_1 > w_2$, let r be the largest integer such that $w_2 = \cdots = w_r = w_1 - 1$. Especially, if $w_1 > w_2 + 1$, we let $r = 1$. Then set $\mathcal{G}^*(\overline{w}) = \{G^*(\overline{w})\} \cup \{G^*_{ij} : (i,j) \in [q-1] \times [q-1] \setminus \{(1,1), (1,2), (1,3), \ldots, (1,r), (2,1), (3,1) \ldots, (r,1)\}\}$.

(ii) When $w_1 = w_2$, let r be the largest integer such that $w_1 = \cdots = w_r$. Then set $\mathcal{G}^*(\overline{w}) = \{G^*(\overline{w})\} \cup \{G^*_{ij} : (i,j) \in [q-1] \times [q-1] \setminus \overline{\binom{[r]}{2}}\}$.

Example 3.6. Let $\overline{w} = [3, 2]$. The edge-colored digraph $G^*(\overline{w})$ is given below, where the lines “ $\longrightarrow$ ”, “ $- - \rightarrow$ ”, “ $\cdots\cdots\rightarrow$ ”, and “ $\rightsquigarrow$ ” denote directed edges with colors $(1, 1)$, $(1, 2)$, $(2, 1)$, and $(2, 2)$ respectively.



Then G^*_{22} is the digraph $y_{22} \rightsquigarrow z_{22}$, and the family of digraphs is given by $\mathcal{G}^*(\overline{w}) = \{G^*(\overline{w}), G^*_{22}\}$.

Construction of an optimal $\mathbf{CCC}(n, 2w-3, \overline{w})_q$. Suppose a superpure $\mathcal{G}^*(\overline{w})$-decomposition of $K_n^{(q-1)^2}$ exists. For F isomorphic to $G^*(\overline{w})$, there is a unique partition of vertex $V(F) = \bigcup_{i=1}^{q-1} S_i$ so that the edges from x to y in F have color (i,j) if $x \in S_i$ and $y \in S_j$. Construct a codeword u with support $V(F)$ such that $\mathsf{u}_x = i$ for $x \in S_i$. So, u has composition $\overline{w}$. Let $\mathcal{C}$ be the code consisting of all the codewords u constructed in this way. Now, we check that $\mathcal{C}$ is the desired code.

Since we have a $\mathcal{G}^*(\overline{w})$-decomposition of $K_n^{(q-1)^2}$, for any $i, j \in [q-1]$, the pairs $(x,y) \in \overline{\binom{[n]}{2}}$ in the edges $((x,y),(i,j))$ with color (i,j) are distinct; that is, the pairs in $\{(x,y) : \mathsf{u}_x = i, \mathsf{u}_y = j, x \neq y,$ for any $\mathsf{u} \in \mathcal{C}\}$ are all distinct by the definition of $G^*(\overline{w})$, and then (C3) of Lemma 3.5 is satisfied. Since the decomposition is superpure, (C4) is also met as in section 3.1. Furthermore, if $w_1 \neq w_2$, since the number of edges of color $(1,1)$ in $G^*(\overline{w})$ is $w_1(w_1-1)$, and the total number of edges of color $(1,1)$ in $K_n^{(q-1)^2}$ is $n(n-1)$, we obtain that $\mathcal{C}$ is a $\mathrm{CCC}(n, 2w-3, \overline{w})_q$ code of size $n(n-1)/(w_1(w_1-1))$ by Lemma 3.5; if $w_1 = w_2$, by counting the number of edges of color $(1,2)$ similarly, we obtain that $\mathcal{C}$ is a $\mathrm{CCC}(n, 2w-3, \overline{w})_q$ code of size $n(n-1)/w_1^2$. The code $\mathcal{C}$ meets the upper bound in (2) and is hence optimal.

Computation of $\boldsymbol{\alpha(\mathcal{G}(\overline{w}))}$ and $\boldsymbol{\beta(\mathcal{G}(\overline{w}))}$.

(i) When $w_1 > w_2$: Recall that r is the largest integer such that $w_2 = \cdots = w_r = w_1 - 1$ and $\mathcal{G}^*(\overline{w}) = \{G^*(\overline{w})\} \cup \{G^*_{ij} : (i,j) \in [q-1] \times [q-1] \setminus \{(1,1),(1,2),(1,3),\ldots,(1,r),(2,1),(3,1)\ldots,(r,1)\}\}$.

Consider the digraph $G^*(\overline{w})$. Observe that for $i, j \in [q-1]$ and $k \in [w_i]$, we have $\mathrm{in}_{(i,i)}(x_{ik}, G^*(\overline{w})) = \mathrm{out}_{(i,i)}(x_{ik}, G^*(\overline{w})) = w_i - 1$, $\mathrm{in}_{(j,i)}(x_{ik}, G^*(\overline{w})) = \mathrm{out}_{(i,j)}(x_{ik}, G^*(\overline{w})) = w_j$ for $j \neq i$, and the other indegrees and outdegrees are zero. For the digraph G^*_{ij}, we have $\mathrm{out}_{(i,j)}(y_{ij}, G^*_{ij}) = \mathrm{in}_{(i,j)}(z_{ij}, G^*_{ij}) = 1$, and the other indegrees and outdegrees are zero.

Consider the vector

$$\upsilon = w_1 \tau(x_{11}, G^*(\overline{w})) + (w_1 - 1) \sum_{i=2}^{q-1} \tau(x_{i1}, G^*(\overline{w})).$$

For $(i,j) \in [q-1] \times [q-1]$, let $\mathrm{in}_{(i,j)}(\upsilon)$ and $\mathrm{out}_{(i,j)}(\upsilon)$ denote the coordinates in υ corresponding to the summation of the indegrees and outdegrees with respect to color

(i, j). Then we have

$$\mathrm{in}_{(i,j)}(v) = \begin{cases} w_1(w_1-1) & \text{for } i=j=1, \\ w_1(w_1-1) & \text{for } i=1, j\neq 1, \\ w_1 w_i & \text{for } i\neq 1, j=1, \\ (w_1-1)(w_i-1) & \text{for } i=j, i\neq 1, \\ w_i(w_1-1) & \text{otherwise}, \end{cases}$$

$$\mathrm{out}_{(i,j)}(v) = \begin{cases} w_1(w_1-1) & \text{for } i=j=1, \\ w_1 w_j & \text{for } i=1, j\neq 1, \\ w_1(w_1-1) & \text{for } i\neq 1, j=1, \\ (w_1-1)(w_i-1) & \text{for } i=j, i\neq 1, \\ w_j(w_1-1) & \text{otherwise}. \end{cases}$$

Observe that the coordinates of v corresponding to indegrees and outdegrees with respect to colors in $\{(1,1),(1,2),(1,3),\ldots,(1,r),(2,1),(3,1)\ldots,(r,1)\}$ have value $w_1(w_1-1)$. All other coordinates have values at most $w_1(w_1-1)$. Adding to v a suitable nonnegative integral combination of $\tau(y_{ij}, G^*_{ij})$'s and $\tau(z_{ij}, G^*_{ij})$'s, we obtain $w_1(w_1-1)\cdot(1,1,\ldots,1)$. Hence, we conclude that $\alpha(\mathcal{G}^*(\overline{w})) \mid (w_1(w_1-1))$.

Next, consider the edge vector $\mu(G)$ with $G \in \mathcal{G}^*(\overline{w})$. For $G^*(\overline{w})$, we have $m_{(i,i)}(G^*(\overline{w})) = w_i(w_i-1)$ and $m_{(i,j)}(G^*(\overline{w})) = w_i w_j$ for $j \neq i$. On the other hand, for G^*_{ij} we have $m_{(i,j)}(G^*_{ij}) = 1$ and $m_{(i',j')}(G^*_{ij}) = 0$ for $(i',j') \neq (i,j)$. Consider the vector

$$v' = \mu(G^*(\overline{w})) + \sum_{(i,j)\in[q-1]\times[q-1]\setminus\{(1,a),(a,1):a\in[r]\}} (w_1(w_1-1) - m_{(i,j)}(G^*(\overline{w})))\mu(G^*_{ij}).$$

Since each coordinate of v' is $w_1(w_1-1)$, we have $\beta(\mathcal{G}^*(\overline{w})) \mid (w_1(w_1-1))$. It is obvious that $\frac{1}{w_1(w_1-1)}v'$ is the all one vector of positive rational combination of $\{\mu(G) : G \in \mathcal{G}^*(\overline{w})\}$; then we have that $\mathcal{G}^*(\overline{w})$ is admissible.

(ii) When $w_1 = w_2$: Recall that r is the largest integer such that $w_1 = \cdots = w_r$ and $\mathcal{G}^*(\overline{w}) = \{G^*(\overline{w})\} \cup \{G_{ij} : (i,j) \in [q-1]\times[q-1] \setminus \overline{\binom{[r]}{2}}\}$.

Here, we consider the vector $v = \sum_{i=1}^{q-1} \tau(x_{i1}, G^*(\overline{w}))$. Then the coordinates of v corresponding to indegrees and outdegrees with respect to colors in $\overline{\binom{[r]}{2}}$ have value w_1. All other coordinates have values at most w_1. Adding to v a suitable nonnegative integral combination of $\tau(y_{ij}, G^*_{ij})$'s and $\tau(z_{ij}, G^*_{ij})$'s, we conclude that $\alpha(\mathcal{G}^*(\overline{w})) \mid w_1$. Similar to the case where $w_1 \neq w_2$, we have $\beta(\mathcal{G}^*(\overline{w})) \mid w_1^2$ and $\mathcal{G}^*(\overline{w})$ is admissible.

Finally, applying Corollary 2.4, we obtain the following proposition.

PROPOSITION 3.7. *Fix $\overline{w}$ and let $w = \sum_{i=1}^{q-1} w_i$. There exists an integer n_0 such that*

$$A_q(n, 2w-3, \overline{w}) = \begin{cases} \frac{n(n-1)}{w_1(w_1-1)} & \text{if } w_1 > w_2, \\ \frac{n(n-1)}{w_1^2} & \text{otherwise} \end{cases}$$

for all $n \geq n_0$ satisfying

(i) $n-1 \equiv 0 \pmod{w_1(w_1-1)}$ *if $w_1 > w_2$, and*

(ii) $n-1 \equiv 0 \pmod{w_1^2}$ *otherwise.*

4. Asymptotic exactness of Johnson-type bounds for constant-weight codes. In this section, we construct infinite families of optimal $\mathrm{CWC}(n,d,w)_q$ for $d = 2w-2$ or $d = 2w-3$ and establish the asymptotic exactness of (3) and (4).

Note that constant-composition codes are special instances of constant-weight codes. Hence, we make slight modifications of the definitions of $G(\overline{w})$ and $G^*(\overline{w})$ in section 3 to form the desired families of digraphs for constant-weight codes. Specifically, consider the following set of compositions:

$$W = \left\{ [w_1, w_2, \ldots, w_{q-1}] : 0 \le w_i \le w \text{ for } i \in [q-1], \sum_{i=1}^{q-1} w_i = w \right\}.$$

Example 4.1. Let $w = 3$ and $q = 3$. Then the set W of compositions is given by $\{[0,3],[1,2],[2,1],[3,0]\}$. Furthermore, a 3-ary word u has weight w if and only u has composition belonging to W.

For every $\overline{w} \in W$, we define $G(\overline{w}) = (V(\overline{w}), C(\overline{w}), E(\overline{w}))$ as in (7) and $G^*(\overline{w}) = (V^*(\overline{w}), C^*(\overline{w}), E^*(\overline{w}))$ as in (8). Observe that we do not require all values in the compositions in W to be positive and the compositions to be monotone decreasing.

Finally, we define the following families of digraphs:

$$\mathcal{G}(w) \triangleq \bigcup_{\overline{w} \in W} G(\overline{w}) \quad \text{and} \quad \mathcal{G}^*(w) \triangleq \bigcup_{\overline{w} \in W} G^*(\overline{w}).$$

Note that unlike the digraph families $\mathcal{G}(\overline{w})$ and $\mathcal{G}^*(\overline{w})$, the above digraph families $\mathcal{G}(w)$ and $\mathcal{G}^*(w)$ do not contain digraphs with single edges.

Similar to constructions in section 3, we have that a superpure $\mathcal{G}(w)$-decomposition of $K_n^{(q-1)}$ and a superpure $\mathcal{G}^*(w)$-decomposition of $K_n^{(q-1)^2}$ yield a $\mathrm{CWC}(n, 2w-2, w)_q$ and a $\mathrm{CWC}(n, 2w-3, w)_q$, respectively. To show this, suppose a superpure $\mathcal{G}(w)$-decomposition of $K_n^{(q-1)}$ exists. For each F isomorphic to $G(\overline{w}) \in \mathcal{G}(w)$, there is a unique partition of the vertex set $V(F) = \bigcup_{i=1}^{q-1} S_i$ (here, $S_i = \varnothing$ if $w_i = 0$), so that all outgoing edges from x in F have color i if and only if $x \in S_i$. Then construct one codeword u with support $V(F)$ such that $\mathsf{u}_x = i$ for $x \in S_i$. Hence, u has constant-weight w. Let $\mathcal{C}$ be the code consisting of all the codewords u constructed in this way. Now, we check that $\mathcal{C}$ is the desired code.

Since we have a superpure $\mathcal{G}(w)$-decomposition of $K_n^{(q-1)}$, similar to section 3.1, the conditions (C1) and (C2) of Lemma 3.1 are satisfied. Therefore, the code $\mathcal{C}$ has distance $2w-2$ by Lemma 3.1. Furthermore, since the total number of directed edges in $K_n^{(q-1)}$ is $(q-1)n(n-1)$, and each digraph in $\mathcal{G}(w)$ uses $w(w-1)$ of them, we obtain that $\mathcal{C}$ is a $\mathrm{CWC}(n, 2w-2, w)_q$ of size $(q-1)n(n-1)/(w(w-1))$.

The case for $\mathrm{CWC}(n, 2w-3, w)_q$ can be proved similarly.

Unfortunately, it is not straightforward to determine $\alpha(\mathcal{G}(w))$ and $\beta(\mathcal{G}(w))$. Instead, we follow the methodology of Lamken and Wilson (see [23, Theorem 8.1]) to complete the proof of Proposition 4.3. Specifically, by Theorem 2.3, to make sure the superpure $\mathcal{G}(w)$-decomposition of $K_n^{(q-1)}$ exists, it is sufficient to show that provided $(q-1)n(n-1) \equiv 0 \pmod{w(w-1)}$, $n-1 \equiv 0 \pmod{w-1}$, we have the following:

(i) $n(n-1) \cdot (1,1,\ldots,1)$ of length $q-1$ is an integral linear combination of vectors in $\{\mu(G) : G \in \mathcal{G}(w)\}$, that is, $n(n-1) \equiv 0 \pmod{\beta(\mathcal{G}(w))}$;

(ii) $(n-1) \cdot (1,1,\ldots,1)$ of length $2(q-1)$ is an integral linear combination of vectors in $\{\tau(u,G) : u \in G, G \in \mathcal{G}(w)\}$, that is, $n-1 \equiv 0 \pmod{\alpha(\mathcal{G}(w))}$; and

(iii) $\mathcal{G}(w)$ is admissible.

To establish (i) and (ii), we apply the following lemma.

LEMMA 4.2 (Schrijver [25]). *Let M be a rational m by n matrix, and let c be a rational vector of length m. Then $Mx = c$ has an integral solution if and only if for all rational vectors y of length m, $y^T c$ is integral whenever $y^T M$ is integral.*

For convenience, we write $a \equiv b$ if $a - b$ is an integer.

Proof of (i). To establish the implication by Lemma 4.2, it suffices to show the following: whenever we are given $q-1$ rationals X_i with $i \in [q-1]$ such that

$$\sum_i X_i m_i(G) \equiv 0 \quad \text{for all } G \in \mathcal{G}(w),$$

then

$$n(n-1)\sum_i X_i \equiv 0.$$

For $i \in [q-1]$, consider the digraph $G(\overline{w})$ with $w_i = w$ and $w_j = 0$ for $j \neq i$. Hence, we have

$$w(w-1)X_i \equiv 0. \tag{9}$$

For $(i,j) \in \overline{\binom{[q-1]}{2}}$, consider the digraph $G(\overline{w})$ with $w_i = w-1$, $w_j = 1$, and $w_k = 0$ for $k \notin \{i,j\}$. Hence, we have

$$(w-1)^2 X_i + (w-1)X_j \equiv 0. \tag{10}$$

Subtracting (10) from (9), we have $(w-1)X_i \equiv (w-1)X_j$ for all i,j. Since $w-1$ divides $n-1$, we have

$$(n-1)X_i \equiv (n-1)X_j.$$

Finally, we have

$$n(n-1)\sum_i X_i \equiv (q-1)n(n-1)X_1 \equiv 0$$

since $w(w-1)$ divides $(q-1)n(n-1)$ and (9) holds. $\square$

Proof of (ii). To establish the implication by Lemma 4.2, it suffices to show the following: whenever we are given $2(q-1)$ rationals X_i, Y_i with $i \in [q-1]$ such that

$$\sum_i X_i \mathrm{in}_i(u,G) + Y_i \mathrm{out}_i(u,G) \equiv 0 \quad \text{for all } G \in \mathcal{G}(w) \text{ and } u \in G,$$

then

$$(n-1)\sum_i (X_i + Y_i) \equiv 0. \tag{11}$$

For $i \in [q-1]$, consider the digraph $G(\overline{w})$ with $w_i = w$ and $w_j = 0$ for $j \neq i$, and consider any vertex in $G(\overline{w})$. Hence, we have $(w-1)X_i + (w-1)Y_i \equiv 0$. Since $(w-1)$ divides $(n-1)$,

$$(n-1)X_i + (n-1)Y_i \equiv 0.$$

Therefore, (11) is immediate. $\square$

Proof of (iii). Summing up $\mu(G)$ for $G \in \mathcal{G}(w)$, we obtain a constant vector of length $q-1$ by symmetry. Therefore, admissibility of $\mathcal{G}(w)$ is immediate. $\square$

PROPOSITION 4.3. *Fix w. There exists an integer n_0 such that*

$$A_q(n, 2w-2, w) = \frac{(q-1)n(n-1)}{w(w-1)}$$

for all $n \geq n_0$ satisfying $(q-1)n(n-1) \equiv 0 \pmod{w(w-1)}$ and $n-1 \equiv 0 \pmod{w-1}$.

On the other hand, $\mathcal{G}^*(w)$ corresponds to the family of digraphs constructed in the proof of [23, Theorem 8.1]. Therefore, we have the following proposition.

PROPOSITION 4.4. *Fix w. There exists an integer n_0 such that*

$$A_q(n, 2w-3, w) = \frac{(q-1)^2 n(n-1)}{w(w-1)}$$

for all $n \geq n_0$ satisfying $(q-1)^2 n(n-1) \equiv 0 \pmod{w(w-1)}$ and $(q-1)(n-1) \equiv 0 \pmod{w-1}$.

5. Asymptotic exactness of Johnson-type bounds for codes with constant composition using group divisible codes. Chee, Ge, and Ling [6] first introduced the notion of group divisible codes (GDCs), and later many authors [17, 35, 37, 36] made use of GDCs to construct many families of optimal CWCs and CCCs. In this section, we construct families of GDCs using the decompositions of edge-colored digraphs and hence obtain families of optimal CCC$(n, 2w-2, \overline{w})$ that are different from those in section 3.

Given a vector $\mathsf{u} = (\mathsf{u}_x)_{x \in X} \in \mathbb{Z}_q^X$ and $Y \subseteq X$, the *restriction of* u *to* Y, denoted by $\mathsf{u}|_Y$, is the vector $\mathsf{v} \in \mathbb{Z}_q^Y$ such that $\mathsf{v} = (\mathsf{u}_x)_{x \in Y}$. Let $|X| = m$, $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a partition of X and $\mathcal{C} \subseteq \mathbb{Z}_q^X$. A q-ary GDC of distance d is a triple $(X, \mathcal{P}, \mathcal{C})$, where $\mathcal{C}$ is a q-ary code of distance d, and $\left\| \mathsf{u}|_{P_i} \right\| \leq 1$ for all $\mathsf{u} \in \mathcal{C}$ and $1 \leq i \leq n$. Elements of the partition $\mathcal{P}$ are called *groups,* and the *type* of GDC $(X, \mathcal{P}, \mathcal{C})$ is the multiset $\{|P| : P \in \mathcal{P}\}$. We are interested in the special case where $|P_1| = |P_2| = \cdots = |P_n| = g$, and we say that the GDC is of type g^n.

If a GDC $(X, \mathcal{P}, \mathcal{C})$ with distance d and length m is of constant-composition $\overline{w}$, we denote $\mathcal{C}$ by GDC$(m, d, \overline{w})$. Here, we construct a family of GDC$(gn, 2w-2, \overline{w})$ of type g^n which yields another class of optimal CCC$(gn, 2w-2, \overline{w})$. To do so, we define the following family of edge-colored digraphs.

Definition of the family $\hat{\mathcal{G}}(\overline{w}, g)$. For a fixed composition $\overline{w}$ and group size g, let $\Gamma(\overline{w}, g)$ denote the following set of $\big((q-1)g\big)$-tuple of nonnegative integers:

$$\Gamma(\overline{w}, g) \triangleq \left\{ \mathsf{t} = (t_{ik})_{i \in [q-1], k \in [g]} : \sum_{k=1}^{g} t_{ik} = w_i \text{ for all } i \in [q-1] \right\}.$$

For each $\mathsf{t} \in \Gamma(\overline{w}, g)$, define an edge-colored digraph $G(\mathsf{t}) = (V(\mathsf{t}), C(\mathsf{t}), E(\mathsf{t}))$, where

$$\begin{aligned} V(\mathsf{t}) &\triangleq \bigcup_{i=1}^{q-1} \bigcup_{k=1}^{g} T_{ik}, \text{ where } T_{ik} = \left\{ x_{ik}^{(r)} : r \in [t_{ik}] \right\}, \\ C(\mathsf{t}) &\triangleq [q-1] \times [g] \times [g], \\ E(\mathsf{t}) &\triangleq \left\{ ((x, x'), (i, k, \ell)) : x \in T_{ik}, x' \in T_{j\ell} \text{ and } x' \neq x \right\}. \end{aligned}$$

In other words, $G(\mathsf{t})$ is a digraph defined on w vertices. Its vertex set $V(\mathsf{t})$ is partitioned into $(q-1)g$ classes, where the size of the class T_{ik} is determined by t_{ik},

$i \in [q-1]$, and $k \in [g]$. Suppose that x belongs to $T_{ik} \subseteq V(\mathsf{t})$. Observe that the edges leaving x necessarily have color $(i,k,*)$. Since every vertex is connected to each of the other $w-1$ vertices and there are $\sum_{k=1}^{g} t_{1k} = w_1$ vertices in $\bigcup_{k=1}^{g} T_{1k}$, there are exactly $w_1(w-1)$ edges of color $(1,*,*)$.

In addition to the graphs $G(\mathsf{t})$, we consider the following graphs. Let s be the largest integer such that $w_1 = w_2 = \cdots = w_s$. For each $s+1 \le i \le q-1$ and $k, \ell \in [g]$, let $G_{ik\ell}$ be an edge-colored digraph with two vertices $y_{ik\ell}$, $z_{ik\ell}$ and one directed edge with color (i,k,ℓ) from $y_{ik\ell}$ to $z_{ik\ell}$.

Finally, we define the family of graphs of interest. Set

$$\hat{\mathcal{G}}(\overline{w}, g) \triangleq \{G(\mathsf{t}) : \mathsf{t} \in \Gamma(\overline{w}, g)\} \cup \{G_{ik\ell} : s+1 \le i \le q-1, k, \ell \in [g]\}.$$

Construction of a GDC$(gn, 2w-2, \overline{w})$ of type g^n. Suppose that a superpure $\hat{\mathcal{G}}(\overline{w}, g)$-decomposition of $K_n^{((q-1)g^2)}$ exists. Then we construct a code $\mathcal{C} \subseteq \mathbb{Z}_q^{[n]\times[g]}$. Specifically, for each F isomorphic to one digraph in $\{G(\mathsf{t}) : \mathsf{t} \in \Gamma(\overline{w}, g)\}$, there is a unique partition of vertex $V(F) = \bigcup_{i=1}^{q-1} \bigcup_{k=1}^{g} T_{ik}$ so that the outgoing edges from x have color $(i,k,*)$ if and only if $x \in T_{ik}$. Then we construct the codeword u with support $V(F)$ by assigning $\mathsf{u}_{(x,k)} = i$ for $x \in T_{ik}$. Let $\mathcal{C}$ be the code consisting of all the codewords u constructed in this way.

Similarly to section 3.1, by checking the conditions (C1) and (C2) in Lemma 3.1, we can see that the code $\mathcal{C}$ has distance $2w-2$. Furthermore, if we set $X = [n] \times [g]$ and $\mathcal{P} = \{\{x\} \times [g] : x \in [n]\}$, we verify that $(X, \mathcal{P}, \mathcal{C})$ is a GDC$(gn, 2w-2, \overline{w})$ of type g^n. Observe that there are $g^2 n(n-1)$ edges with color $(1,*,*)$ in $K_n^{((q-1)g^2)}$, and each digraph in $\{G(\mathsf{t}) : \mathsf{t} \in \Gamma(\overline{w}, g)\}$ uses $w_1(w-1)$ of them. Therefore, the size of $\mathcal{C}$ is given by $g^2 n(n-1)/(w_1(w-1))$.

To demonstrate the existence of such decompositions, we use a method similar to that in section 4. As the proof is tedious and analogous, we defer it to the appendix.

PROPOSITION 5.1. *Let $g \ge 2$ and $w_1 \ge 2$. For each $\overline{w} = [w_1, w_2, \ldots, w_{q-1}]$, there exists n_0 such that a $\overline{w}$-GDC$(2w-2)$ of type g^n and size $g^2 n(n-1)/(w_1(w-1))$ exists for all $n \ge n_0$ satisfying*

$$\begin{aligned} gn(n-1) &\equiv 0 \pmod{w_1(w-1)}, \\ g(n-1) &\equiv 0 \pmod{w-1}. \end{aligned}$$

Taking $g = w-1$ in Proposition 5.1, we obtain a CCC$((w-1)n, 2w-2, \overline{w})$ of size $(w-1)n(n-1)/w_1$. These codes are optimal since (1) provides the upper bound

$$\left\lfloor \frac{(w-1)n}{w_1} \left\lfloor \frac{(w-1)n-1}{w-1} \right\rfloor \right\rfloor = \frac{(w-1)^2 n(n-1)}{w_1(w-1)} = \frac{(w-1)n(n-1)}{w_1}.$$

Hence, we obtain the following corollary.

COROLLARY 5.2. *For each $\overline{w} = [w_1, w_2, \ldots, w_{q-1}]$ with $w_1 \ge 2$, there exists an integer n_0 such that*

$$A_q((w-1)n, 2w-2, \overline{w}) = \frac{(w-1)n(n-1)}{w_1}$$

for all $n \ge n_0$ satisfying $n(n-1) \equiv 0 \pmod{w_1}$.

Finally, we remark that the family of optimal codes in Corollary 5.2 is different from that in Proposition 3.4. Specifically, the length of codes in Corollary 5.2 is a multiple of $w-1$, while the codes in Proposition 3.4 have lengths congruent to 1 modulo $w-1$.

6. Asymptotic exactness of bounds for multiply constant-weight codes. In this section, we consider the asymptotic exactness of the bounds in Lemma 2.2 for multiply constant-weight codes using two different methods.

6.1. When distance $d = 2mw - 2$. First, we construct an infinite family of optimal MCWC$(m, n, 2mw-2, w)$ that meets the upper bound (5). Fix m and w.

Using digraphs from section 3.2, we define the edge-colored digraph $H^*(m, w) \triangleq G^*(\overline{mw})$, where $w_1 = w_2 = \cdots = w_m = w$ (here, $q - 1 = m$). For $i \in [m]$, let G^*_{ii} be a digraph with vertices y_{ii}, z_{ii} and one directed edge with color (i, i) from y_{ii} to z_{ii}. Finally, define $\mathcal{H}^*(m, w) = \{H^*(m, w)\} \cup \{G^*_{ii} : i \in [m]\}$.

Construction of an optimal MCWC$(m, n, 2mw - 2, w)$. Suppose an $\mathcal{H}^*(m, w)$-decomposition of $K_n^{(m^2)}$ exists. For F isomorphic to $H^*(m, w)$, there is a unique partition of vertex $V(F) = \bigcup_{i=1}^m S_i$ so that the edges from x to y in F have color (i, j) if and only if $x \in S_i$ and $y \in S_j$. Then construct one codeword u such that $\mathsf{u}_{(i,x)} = 1$ for $i \in [m]$ and $x \in S_i$. Let $\mathcal{C}$ be the code consisting of all the codewords u constructed in this way.

Since we have an $\mathcal{H}^*(m, w)$-decomposition of $K_n^{(m^2)}$, for any $\mathsf{u}, \mathsf{v} \in \mathcal{C}$, $|\mathrm{supp}(\mathsf{u}) \cap \mathrm{supp}(\mathsf{v})| \leq 1$, which means that the distance of the code is at least $2mw - 2$. Furthermore, since the number of edges in $K_n^{(m^2)}$ of color $(1, 2)$ is $n(n-1)$, and each digraph isomorphic to $H^*(m, w)$ uses w^2 of them, we have an MCWC$(m, n, 2mw - 2, w)$ of size $n(n-1)/w^2$ that meets the upper bound given in Lemma 2.2. As $H^*(m, w) = G^*(\overline{mw})$, computations similar to those in section 3.2 yield $\alpha(\mathcal{H}^*(m, w)) \mid w$, $\beta(\mathcal{H}^*(m, w)) \mid w^2$, and $\mathcal{H}^*(m, w)$ is admissible. Applying Corollary 2.4, we obtain the following.

Proposition 6.1. *Fix m and w. There exists an integer n_0 so that*

$$M(m, n, 2mw - 2, w) = \frac{n(n-1)}{w^2}$$

for all $n \geq n_0$ satisfying $n(n-1) \equiv 0 \pmod{w^2}$ and $n-1 \equiv 0 \pmod{w}$, i.e., $n-1 \equiv 0 \pmod{w^2}$.

6.2. When distance $d = 2(mw - \lambda)$. We consider the optimal multiply constant-weight codes reaching the bound (6). First, we need the following concepts from design theory.

For any positive integers v, k, λ, a *balanced incomplete block design,* (v, k, λ)-BIBD, is a decomposition of λK_v (that is, each edge in K_v appears λ times) into K_k. The K_k's in the decomposition are called *blocks.* A (v, k, λ)-BIBD is said to be *α-resolvable* if the blocks can be partitioned into classes $R_1, R_2, \ldots, R_r$ (called *parallel classes*) where $r = \lambda(v-1)/(\alpha(k-1))$ such that each vertex of K_v is contained in precisely α blocks of each class.

Wang et al. [30] established the following relation between α-resolvable BIBDs and multiply constant-weight codes.

Proposition 6.2 (Wang et al. [30]). *If there exists an α-resolvable (v, k, λ)-BIBD, then $M(m, n, d, w) = v$, where $m = \lambda(v-1)/(\alpha(k-1))$, $n = \alpha v/k$, $d = 2(mw - \lambda) = 2\lambda(v-k)/(k-1)$, and $w = \alpha$.*

Using this relation, Wang et al. then determined the value of $M(m, n, 2(mw - 2), w)$ for sufficiently large m.

Theorem 6.3 (Wang et al. [30]). *Given positive integers k and w with $k|w$, there* *exists a constant* $m_0 = m_0(k,w)$ *such that*

$$M(m,n,2(mw-w),w) = m(k-1)+1$$

with $n = w(m(k-1)+1)/k$ *for some* $m \geq m_0$.

Recently, Dukes, Ling, and Malloch [15] established the following asymptotic existence of α-resolvable BIBDs.

THEOREM 6.4. *Let* k, α, λ *be positive integers, and let* $k \geq 2$. *The* α*-resolvable* (v,k,λ)*-BIBD exists if and only if* $\alpha v \equiv 0 \pmod{k}$ *and* $\lambda(v-1) \equiv 0 \pmod{\alpha(k-1)}$ *for sufficiently large* v.

Combining Proposition 6.2 and Theorem 6.4, we have the following result.

THEOREM 6.5. *Given positive integers* k, w, *and* $\lambda < w$, *there exists a constant* $m_0 = m_0(k,w,\lambda)$ *such that*

$$M(m,n,2(mw-\lambda),w) = \frac{mw(k-1)}{\lambda} + 1$$

with $n = w/k\,(1 + mw(k-1)/\lambda)$ *for some* $m \geq m_0$.

7. Conclusion. We verify that some bounds, in particular, the Johnson bounds, are asymptotically exact for several generalizations of binary constant-weight codes. This was achieved via an interesting application of superpure decompositions of edge-colored digraphs. For easy reference, in Table 1 we summarize the families of digraphs used in this paper and their corresponding codes.

TABLE 1
Families of edge-colored digraphs and their corresponding codes.

Family of digraphs	Codes	Remarks
$\mathcal{G}(\overline{w})$	$\mathrm{CCC}(n,2w-2,\overline{w})_q$	Proposition 3.4
$\mathcal{G}^*(\overline{w})$	$\mathrm{CCC}(n,2w-3,\overline{w})_q$	Proposition 3.7
$\mathcal{G}(w)$	$\mathrm{CWC}(n,2w-2,w)_q$	Proposition 4.3
$\hat{\mathcal{G}}(\overline{w},g)$	$\mathrm{GDC}(gn,2w-2,\overline{w})$ of type g^n	Proposition 5.1
$\mathcal{H}^*(m,w)$	$\mathrm{MCWC}(m,n,2mw-2,w)$	Proposition 6.1

Observe that in Propositions 3.4, 3.7, 4.3, 4.4, 5.1, and 6.1, the Johnson-type bounds are shown to be exact for sufficiently large n satisfying certain congruence conditions. We hypothesize that the Johnson-type bounds are tight up to an additive constant for sufficiently large n. Specifically, we make the following conjecture.

Conjecture. Fix q, $\overline{w}$, w, m, and let $d \in \{2w-2, 2w-3\}$. Define $U_q(n,d,\overline{w})$, $U_q(n,d,w)$, and $U(m,n,d,w)$ to be the upper bounds given by (1)–(5). Then

$$\begin{aligned} A_q(n,d,\overline{w}) &= U_q(n,d,\overline{w}) - O(1),\\ A_q(n,d,w) &= U_q(n,d,w) - O(1),\\ M(m,n,2mw-2,w) &= U(m,n,2mw-2,w) - O(1). \end{aligned}$$

The case for binary constant-weight codes, that is, $q = 2$ and $d = 2w-2$, has been verified by Chee et al. [2].

Finally, we point to recent remarkable results in design theory. Keevash [21] established the asymptotic exactness of the Johnson bound for fixed weight w and distance d. This implies that the Johnson bound is asymptotically exact for the classes of codes studied in this paper. Another proof of the same result was later provided by Glock et al. [18, 19]. More recently, Keevash [22] extended his techniques to demonstrate asymptotic existence of decompositions of edge-colored digraphs and hypergraphs.

Appendix. On $\hat{\mathcal{G}}(\overline{w},g)$ and the proof of Proposition 5.1. As with $\mathcal{G}(w)$ in section 4, it is difficult to determine $\alpha(\hat{\mathcal{G}}(\overline{w},g))$ and $\beta(\hat{\mathcal{G}}(\overline{w},g))$ exactly. Hence, we adapt the methods of Lamken and Wilson (see [23, Theorem 8.1]) to complete the proof of Proposition 5.1.

First, we make certain observations on the edge vectors and degree vectors of the digraphs in $\hat{\mathcal{G}}(\overline{w},g)$. Recall the definition of $\Gamma(\overline{w},g)$ and set $t_k \triangleq \sum_{i=1}^{q-1} t_{ik}$.

(A) For each $G(\mathsf{t})$ with $\mathsf{t} \in \Gamma(\overline{w},g)$, the edge vector

$$\mu(G(\mathsf{t})) = (m_{(i,k,\ell)} : i \in [q-1], k,\ell \in [g])$$

is of length $(q-1)g^2$. In particular, $m_{(i,k,k)} = t_{ik}(t_k - 1)$ and $m_{(i,k,\ell)} = t_{ik}t_\ell$ for $\ell \neq k$.

For each $u \in V(\mathsf{t})$, the degree vector

$$\tau(u, G(\mathsf{t})) = (\mathrm{in}_{(i,k,\ell)}(u, G(\mathsf{t})), \mathrm{out}_{(i,k,\ell)}(u, G(\mathsf{t})) : i \in [q-1], k,\ell \in [g])$$

is of length $2(q-1)g^2$. Suppose $u \in T_{ik}$ for some $i \in [q-1]$, $k \in [g]$. Then the entries of $\tau(u, G(\mathsf{t}))$ are as follows:

$$\mathrm{in}_{(j,\ell,\ell')}(u, G(\mathsf{t})) = \begin{cases} t_{ik} - 1 & \text{if } (j,\ell) = (i,k),\ \ell' = k, \\ t_{j\ell} & \text{if } (j,\ell) \neq (i,k),\ \ell' = k, \\ 0 & \text{if } \ell' \neq k \end{cases}$$

and

$$\mathrm{out}_{(j,\ell,\ell')}(u, G(\mathsf{t})) = \begin{cases} t_k - 1 & \text{if } (j,\ell) = (i,k),\ \ell' = k, \\ t_{\ell'} & \text{if } (j,\ell) = (i,k),\ \ell' \neq \ell, \\ 0 & \text{if } (j,\ell) \neq (i,k). \end{cases}$$

(B) Recall that s is the largest integer such that $w_1 = w_2 = \cdots = w_s$. For $s+1 \le i \le q-1$, $k,\ell \in [g]$, recall that $G_{ik\ell}$ is the edge-colored digraph with two vertices $y_{ik\ell}$, $z_{ik\ell}$ and one directed edge with color (i,k,ℓ) from $y_{ik\ell}$ to $z_{ik\ell}$.

Therefore, the edge vector $\mu(G_{ik\ell}) = (m_{(i',k',\ell')} : i' \in [q-1], k',\ell' \in [g])$ has entry one at index (i,k,ℓ) and zero otherwise. Similarly, the degree vectors for the vertices $y_{ik\ell}$ and $z_{ik\ell}$ have $\mathrm{in}_{(i,k,\ell)}(z_{ik\ell}, G_{ik\ell}) = \mathrm{out}_{(i,k,\ell)}(y_{ik\ell}, G_{ik\ell}) = 1$ and zeros at all other entries.

Proof strategy. To complete the proof of Proposition 5.1 using Theorem 2.3, it suffices to show that provided $gn(n-1) \equiv 0 \pmod{w_1(w-1)}$ and $g(n-1) \equiv 0 \pmod{w-1}$, we have the following implications:

(i) The vector $n(n-1)\cdot(1,1,\dots,1)$ of length $(q-1)g^2$ is an integral linear combination of the vectors in $\{\mu(G) : G \in \hat{\mathcal{G}}(\overline{w},g)\}$.

(ii) The vector $(n-1)\cdot(1,1,\dots,1)$ of length $2(q-1)g^2$ is an integral linear combination of the vectors in $\{\tau(u,G) : u \in G, G \in \hat{\mathcal{G}}(\overline{w},g)\}$.

(iii) $\hat{\mathcal{G}}(\overline{w},g)$ is admissible.

Proof of (i). As before, we establish the first implication using Lemma 4.2. Let X be a rational column vector of length $(q-1)g^2$ indexed by the color set $[q-1]\times[g]\times[g]$. Let M be a matrix $(q-1)g^2 \times |\hat{\mathcal{G}}(\overline{w},g)|$, where each column of M is an edge vector of a digraph in $\hat{\mathcal{G}}(\overline{w}, g)$. To apply Lemma 4.2, it suffices to show that whenever $X^T M$ is integral, then

$$X^T \cdot n(n-1) \cdot (1,1,\ldots,1)^T = n(n-1) \sum_{i,k,\ell} X_{ik\ell} \equiv 0. \tag{12}$$

Recall that $a \equiv b$ means that $a - b$ is an integer.

For $s+1 \le i \le q-1$, $k, \ell \in [g]$, we have that $\mu(G_{ik\ell})$ has one at the entry indexed by $ik\ell$ and zero at all other entries. Hence, since $X^T M$ is integral, we have $X_{ik\ell} \equiv 0$, and it remains to consider the sum $n(n-1)\sum_{i,k,\ell} X_{ik\ell}$ for $i \in [s]$. Equivalently, we consider the case where $s = q-1$, or $w_i = w_1$ for all $i \in [q-1]$.

Consider $G(\mathsf{t})$ for any $\mathsf{t} \in \Gamma(\overline{w}, g)$. From the characterization of edge vectors, we have that

$$X^T \cdot \mu(G(\mathsf{t}))^T = \sum_i \left[\sum_{k \neq \ell} t_{ik} t_\ell X_{ik\ell} + \sum_k t_{ik}(t_k - 1) X_{ikk} \right] \equiv 0. \tag{13}$$

In what follows, we select t from $\Gamma(\overline{w}, g)$ and establish certain integral equations that are implied from (13). Then using these integral equations, we establish the integral equation (12).

Fix $k \in [g]$. We take the sequence t with $t_{ik} = w_i$ for all $i \in [q-1]$, and all other entries of t are zero. Then we have

$$\sum_i [w_i(w-1)X_{ikk}] \equiv 0. \tag{14}$$

Next, fix three indices $i \in [q-1]$ and $k, \ell \in [g]$, $k \neq \ell$, and take the sequence t with $t_{ik} = w_i - 1$, $t_{i\ell} = 1$, $t_{jk} = w_j$ for any $j \in [q-1] \setminus \{i\}$, and all other entries of t are zero. Then

$$\begin{aligned} \sum_{j \neq i} \Big[w_j X_{jk\ell} + w_j(w-2) X_{jkk} \Big] & \\ + (w_i - 1) X_{ik\ell} + (w-1) X_{i\ell k} &+ (w_i - 1)(w-2) X_{ikk} \equiv 0. \end{aligned} \tag{15}$$

Finally, we fix three indices $i \in [q-1]$ and $k, \ell \in [g]$, $k \neq \ell$, and take the sequence t with $t_{ik} = w_i - 2$ and $t_{i\ell} = 2$. Again, we set $t_{jk} = w_j$ for any $j \in [q-1] \setminus \{i\}$, and all other entries of t are zero. Then

$$\begin{aligned} \sum_{j \neq i} \Big[2w_j X_{jk\ell} + w_j(w-3) X_{jkk} \Big] & \\ + 2(w_i - 2) X_{ik\ell} + 2(w-2) X_{i\ell k} &+ (w_i - 2)(w-3) X_{ikk} + 2X_{i\ell\ell} \equiv 0. \end{aligned} \tag{16}$$

Taking $2 \times (15) - (14) - (16)$, we have that

$$2(X_{ik\ell} + X_{i\ell k}) \equiv 2(X_{i\ell\ell} + X_{ikk}) \quad \text{for any } i \in [q-1],\ k, \ell \in [g],\ k \neq \ell.$$

Taking the summation over i, k, ℓ and multiplying both sides with $n(n-1)/2 \in \mathbb{Z}$, we have

$$n(n-1) \sum_{i,k,\ell} X_{ik\ell} \equiv gn(n-1) \sum_{i,k} X_{ikk}.$$

Suppose that $gn(n-1) \equiv 0 \pmod{w_1(w-1)}$. We have $gn(n-1) = w_1(w-1)n^*$ for some $n^* \in \mathbb{Z}$. Therefore by (14),

$$n(n-1)\sum_{i,k,\ell} X_{ik\ell} \equiv n^* \sum_k \sum_i w_1(w-1)X_{ikk} \equiv 0.$$

Proof of (ii). As before, we establish the first implication using Lemma 4.2. Let $X = (X_{ik\ell}, Y_{ik\ell} : i \in [q-1], k, \ell \in [g])$ be a rational column vector of length $2(q-1)g^2$. Let M be a matrix where each column of M is a degree vector of some vertex of a digraph in $\hat{\mathcal{G}}(\overline{w}, g)$. To apply Lemma 4.2, it suffices to show that whenever $X^T M$ is integral, then

$$X^T \cdot (n-1) \cdot (1,1,\ldots,1)^T = (n-1)\sum_{i,k,\ell}(X_{ik\ell} + Y_{ik\ell}) \equiv 0. \tag{17}$$

Similarly to the proof of (i), we assume that $w_1 = w_2 = \cdots = w_{q-1}$. Consider the digraph $G(\mathsf{t})$ for some $\mathsf{t} \in \Gamma(\overline{w}, g)$, and consider the degree vector for some vertex $u \in T_{ik}$ for some $i \in [q-1]$ and $k \in [g]$. From our characterization of degree vectors, we have that

$$X^T \cdot \tau(u, G(\mathsf{t}))^T = (t_{ik}-1)X_{ikk} + (t_k-1)Y_{ikk} + \sum_{(j,\ell)\neq(i,k)} t_{j\ell}X_{j\ell k} + \sum_{\ell\neq k} t_\ell Y_{ik\ell} \equiv 0. \tag{18}$$

Again, we select t from $\Gamma(\overline{w}, g)$ and establish certain integral equations using (18). Then using these integral equations, we establish the integral equation (17).

First, we take the sequence t to be $t_{jk} = w_j$ for any $j \in [q-1]$, and $t_{j\ell} = 0$ for all $\ell \neq k$. Then we have

$$\sum_j w_j X_{jkk} - X_{ikk} + (w-1)Y_{ikk} \equiv 0. \tag{19}$$

Now fix another index ℓ, $\ell \neq k$, and take the sequence t with $t_{ik} = w_i - 1$, $t_{i\ell} = 1$, and $t_{jk} = w_j$ for any $j \in [q-1]$, $j \neq i$, and the other entries are all zero. Then we have

$$\sum_j w_j X_{jkk} - 2X_{ikk} + X_{i\ell k} + (w-2)Y_{ikk} + Y_{ik\ell} \equiv 0. \tag{20}$$

Taking (19)–(20), we have that

$$X_{i\ell k} + Y_{ik\ell} \equiv X_{ikk} + Y_{ikk} \quad \text{for any } i \in [q-1],\ k, \ell \in [g],\ k \neq \ell\ .$$

Summing on both sides over i, k, ℓ, we have

$$\sum_{i,k,\ell}(X_{i\ell k} + Y_{ik\ell}) \equiv g\sum_{i,k}(X_{ikk} + Y_{ikk}).$$

Summing over i for (19),

$$\sum_i (X_{ikk} + Y_{ikk}) \equiv (q-1)\sum_j w_j X_{jkk} + w\sum_i Y_{ikk} \equiv w\sum_i (X_{ikk} + Y_{ikk}).$$

Hence,

$$(w-1)\sum_i (X_{ikk} + Y_{ikk}) \equiv 0.$$

Suppose $g(n-1) \equiv 0 \pmod{w-1}$ and $g(n-1) = n^*(w-1)$ for some $n^* \in \mathbb{Z}$; then we have

$$\begin{aligned}(n-1)\sum_{i,k,\ell}(X_{ik\ell}+Y_{ik\ell}) &= (n-1)\sum_{i,k,\ell}(X_{ik\ell}+Y_{i\ell k})\\ &\equiv (n-1)g\sum_{i,k}(X_{ikk}+Y_{ikk})\\ &\equiv n^*\sum_{k}(w-1)\sum_{i}(X_{ikk}+Y_{ikk}) \equiv 0.\end{aligned}$$

Proof of (iii). Again, we assume $w_1 = w_2 = \cdots = w_{q-1}$. Let m_1 denote the sum of $\mu(G(\mathsf{t}))$ for all $\mathsf{t} \in \Gamma(\overline{w}, g)$. Observe that from the symmetry of $G(\mathsf{t})$, we can assume the entries of vector m_1 at indices (i,k,k) are A for $i \in [q-1], k \in [g]$, and the entries at indices (i,k,ℓ) are B for $i \in [q-1], k, \ell \in [g]$, and $k \neq \ell$.

Let m_2 denote the sum of $\mu(G(\mathsf{t}))$ for all t with $t_{ik} = w_i$ for all $i \in [q-1]$, and some fixed $k \in [g]$, and all other entries zero. Again from symmetry, we can assume that the entries of the vector m_2 at indices (i,k,k) are C for $i \in [q-1]$, $k \in [g]$. Furthermore, the entries at the other indices are zero.

Let m_3 denote the sum of $\mu(G(\mathsf{t}))$ for all t where $t_{ik} \in \{\lfloor w_1/g \rfloor, \lceil w_1/g \rceil\}$ for $i \in [q-1]$ and $k \in [g]$. Again from symmetry, we can assume that the entries of vector m_3 at indices (i,k,k) are D for $i \in [q-1], k \in [g]$ and the entries at indices (i,k,ℓ) are E for $i \in [q-1], k, \ell \in [g]$, and $k \neq \ell$. We can show that $D < E$.

To show that $\hat{\mathcal{G}}(\overline{w}, g)$ is admissible, it suffices to demonstrate that we can obtain a constant vector via a positive rational combination of edge vectors in $\hat{\mathcal{G}}(\overline{w}, g)$. We have the following cases:

- If $A = B$, we simply take m_1.
- If $A < B$, we add a nonnegative multiple of m_2 to m_1.
- if $A > B$, we add a nonnegative multiple of m_3 to m_1.

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