

Domination Mappings into the Hamming Ball: Existence, Constructions, and Algorithms

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Abstract

The Hamming ball of radius w in $\{0, 1\}^n$ is the set $\mathcal{B}(n, w)$ of all binary words of length n and Hamming weight at most w . We consider injective mappings $\varphi: \{0, 1\}^m \rightarrow \mathcal{B}(n, w)$ with the following *domination property*: every position $j \in [n]$ is dominated by some position $i \in [m]$, in the sense that “switching off” position i in $x \in \{0, 1\}^m$ necessarily switches off position j in its image $\varphi(x)$. This property may be described more precisely in terms of a bipartite *domination graph* $G = ([m] \cup [n], E)$ with no isolated vertices; for all $(i, j) \in E$ and all $x \in \{0, 1\}^m$, we require that $x_i = 0$ implies $y_j = 0$, where $y = \varphi(x)$. Although such domination mappings recently found applications in the context of coding for high-performance interconnects, to the best of our knowledge, they were not previously studied.

In this paper, we begin with simple necessary conditions for the existence of an (m, n, w) -*domination mapping* $\varphi: \{0, 1\}^m \rightarrow \mathcal{B}(n, w)$. We then provide several explicit constructions of such mappings, which show that the necessary conditions are also sufficient when $w = 1$, when $w = 2$ and m is odd, or when $m \leq 3w$. One of our main results herein is a proof that the trivial necessary condition $|\mathcal{B}(n, w)| \geq 2^m$ is, in fact, sufficient for the existence of an (m, n, w) -domination mapping whenever m is sufficiently large. We also present a polynomial-time algorithm that, given any m, n , and w , determines whether an (m, n, w) -domination mapping exists for a domination graph with an equitable degree distribution.

1. Introduction

Given a binary word $\mathbf{y} = (y_1, y_2, \dots, y_n)$, the *Hamming weight of \mathbf{y}* is the number of nonzero positions in \mathbf{y} . Explicitly, $\text{wt}(\mathbf{y}) \stackrel{\text{def}}{=} |\{j \in [n] : y_j \neq 0\}|$. The *Hamming ball of radius w in $\{0, 1\}^n$* is the set $\mathcal{B}(n, w)$ of all words of weight at most w . Explicitly, $\mathcal{B}(n, w) \stackrel{\text{def}}{=} \{\mathbf{y} \in \{0, 1\}^n : \text{wt}(\mathbf{y}) \leq w\}$. Given $m \leq n$, we are interested in injective mappings φ from $\{0, 1\}^m$ into $\mathcal{B}(n, w)$ that establish a certain domination relationship between positions in $x \in \{0, 1\}^m$ and positions in its image $\mathbf{y} = \varphi(x)$. Specifically, one should be able to “switch off” every position $j \in [n]$ in \mathbf{y} (that is, ensure that $y_j = 0$) by switching off a corresponding position $i \in [m]$ in x (that is, setting $x_i = 0$). More precisely, let $G = ([m] \cup [n], E)$ be a bipartite graph with m left vertices and n right vertices. If G has no isolated right vertices, we refer to G as a *domination graph*.

Definition 1. Given an injective map $\varphi: \{0, 1\}^m \rightarrow \mathcal{B}(n, w)$ and a graph $G = ([m] \cup [n], E)$, we say that φ is a *G-domination mapping*, or *G-dominating* in brief, if

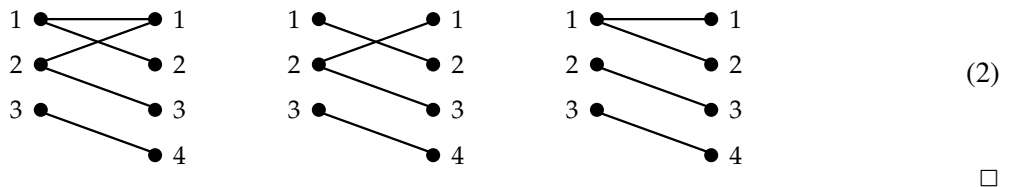
$$\forall (x_1, x_2, \dots, x_m) \in \{0, 1\}^m, \forall (i, j) \in E : \\ \text{if } \varphi(x_1, x_2, \dots, x_m) = (y_1, y_2, \dots, y_n) \text{ and } x_i = 0, \text{ then } y_j = 0$$

We say that φ is an *(m, n, w)-domination mapping* if there exists a domination graph $G = ([m] \cup [n], E)$, with no isolated right vertices, such that φ is *G-dominating*.

Example 1. Let $m = 3$, $n = 4$, $w = 2$, and consider the injective map $\varphi: \{0, 1\}^3 \rightarrow \mathcal{B}(4, 2)$ given by

$$\begin{array}{llll} \varphi(000) = 0000 & \varphi(010) = 0010 & \varphi(100) = 0100 & \varphi(110) = 1000 \\ \varphi(001) = 0001 & \varphi(011) = 0011 & \varphi(101) = 0101 & \varphi(111) = 1001 \end{array} \quad (1)$$

It is easy to verify by inspection that φ is *G-dominating*, where $G = ([3] \cup [4], E)$ can be chosen as any one of the following three domination graphs:



Note that the requirement of no isolated right vertices in Definition 1 means that *every* position $j \in [n]$ in the image of φ is dominated by some position $i \in [m]$ of its domain. This is motivated by the application to thermal-management coding for high-performance interconnects [2, 3], briefly described in what follows.

1.1. Applications to coding for interconnects

Given an interconnect (bus) consisting of n wires, it is desirable to control its average power consumption. This can be achieved by making sure that, during each synchronized transmission, the total number of transitions on the n wires is below a specified threshold w . As shown in [2], the above translates directly into the requirement that the transmitted binary word \mathbf{y} is in the Hamming ball $\mathcal{B}(n, w)$. It is also desirable to control the peak temperature of an interconnect by effectively cooling its hottest wires. That is, given an arbitrary subset S of $[n]$, which corresponds to the positions of the hottest wires, we wish to transmit a word (y_1, y_2, \dots, y_n) such that $y_j = 0$ for all $j \in S$. The *cooling codes* of [2] provide an elegant solution to this problem using spreads or partial spreads [4, 12]. Domination mappings were introduced in [3] in order to simultaneously achieve both of these desirable properties. Given a subset S of $[n]$, one can follow the edges of

the domination graph G to find a subset S' of $[m]$ so that every position in S is dominated by some position in S' . Then, using spreads as described in [2], one can produce a binary word $x = (x_1, x_2, \dots, x_m)$ such that $x_i = 0$ for all $i \in S'$. If φ is a G -dominating mapping from $\{0, 1\}^m$ into $\mathcal{B}(n, w)$, the transmitted word $y = \varphi(x)$ satisfies $\text{wt}(y) \leq w$ and $y_j = 0$ for all $j \in S$. Since φ is injective, x can be recovered from y at the receiver. For more details on this encoding/decoding procedure, we refer the reader to [2, 3].

In the context of coding for high-performance interconnects, the parameters m and w would be usually given, and one would like to use an (m, n, w) -domination mapping with the lowest possible n , in order to minimize the redundancy $n - m$ of the encoding procedure. We note that although domination mappings were defined in [3], they were not studied therein. In particular, it is not clear when such mappings exist.

1.2. Our contributions

Our primary objective in this paper is to answer the following question: for which parameters m, n, w , do (m, n, w) -domination mappings exist? A secondary objective is to study the structure of such mappings and provide explicit constructions for them. We believe that, aside from their applications to coding for interconnects, these problems are also interesting in their own right, from the combinatorial perspective.

One of our main results is the following theorem, which shows that if the domain $\{0, 1\}^m$ is sufficiently large with respect to w , then domination mappings *always* exist.

Theorem 1. *For every $w \geq 1$, there is a constant $m_0(w)$ with the following property: for all $m \geq m_0(w)$, if $|\mathcal{B}(n, w)| \geq 2^m$, then an (m, n, w) -domination mapping exists.*

Thus, for sufficiently large m , whenever injections from $\{0, 1\}^m$ into $\mathcal{B}(n, w)$ exist at all, one of these injections is necessarily an (m, n, w) -domination mapping. This is somewhat surprising; it is certainly not true for small m . To prove Theorem 1, we introduce a certain bipartite graph with vertex set $\{0, 1\}^m \cup \mathcal{B}(n, w)$, which we call the *compatibility graph*. We then use Hall's marriage theorem [5] to show that the compatibility graph has a perfect matching, when m is sufficiently large. The fact that the conditions of Hall's marriage theorem are satisfied in this graph is established in a long and elaborate sequence of lemmas in Section 6.

The other contributions in this paper are organized as follows. We begin in the next section by establishing some basic properties of domination graphs and domination mappings. In particular, we consider certain restrictions of a domination mapping $\varphi: \{0, 1\}^m \rightarrow \mathcal{B}(n, w)$ to a subset of the m positions of its domain. We will show that all such restrictions are also domination mappings. In Section 3, we use some of these results to derive necessary conditions for the existence of an (m, n, w) -domination mapping. In particular, we show that $n \geq 2m - w$ in any such mapping. In Section 4, we present several explicit constructions of domination mappings. In particular, we introduce the product construction; given any two (m_1, n_1, w_1) -domination and (m_2, n_2, w_2) -domination mappings, this construction produces an $(m_1 + m_2, n_1 + n_2, w_1 + w_2)$ -domination mapping. We also construct *perfect* $(m, n, 1)$ -domination mappings with $n = 2^m - 1$, so that $|\mathcal{B}(n, 1)| = 2^m$ and φ is a bijection. We furthermore describe another construction for $w = 2$ and m odd; it appears that already in this case, the construction problem is far from trivial. In Section 5, we present a polynomial-time algorithm that, given any m, n , and w , determines whether an (m, n, w) -domination mapping exists for a domination graph G with an equitable degree distribution (that is, the degree of all the left vertices in G is either $\lfloor n/m \rfloor$ or $\lceil n/m \rceil$). To this end, we apply the König-Egeváry theorem [7] to the bipartite compatibility graph in order to conclude that the maximum matching and the *maximum fractional matching* in this graph have the same size. This leads to a linear program with an exponential number of variables, but we use symmetries of the compatibility graph to show that this problem can be solved in polynomial time.

One consequence of our constructions in Section 4 is that the necessary conditions in Section 3 are also sufficient for all $m \leq 3w$. This resolves the existence problem for (m, n, w) -domination mappings for small

values of m . As a consequence of Theorem 1, this problem is also resolved for large m . For intermediate values of m , we can determine whether an (m, n, w) -domination mapping exists in polynomial time using the algorithm of Section 5 — albeit only for an equitable domination graph. We believe, however, that the decision produced by the polynomial-time algorithm of Section 5 holds unconditionally (cf. Conjecture 1).

2. Basic Properties of Domination Mappings

We begin with some basic properties and notation for domination graphs. Recall that, by definition, a domination graph is a bipartite graph $G = ([m] \cup [n], E)$ with no isolated vertices on the right side of the bipartition. The following lemma shows that G cannot have isolated vertices on the left side either.

Lemma 2. *Let $G = ([m] \cup [n], E)$ be a domination graph. If there is a vertex of degree zero in $[m]$, then a G -domination mapping does not exist.*

Proof. Assume to the contrary that $\varphi: \{0, 1\}^m \rightarrow \mathcal{B}(n, w)$ is a G -domination mapping. Since every position in $[n]$ is dominated by some position in $[m]$, we have $\varphi(\mathbf{0}) = \mathbf{0}$, where $\mathbf{0}$ denotes the all-zero word of the appropriate length. Suppose that the vertex $i \in [m]$ has degree zero, and let $e_i \in \{0, 1\}^m$ be the word of weight one with the single nonzero in position i . Then $\varphi(e_i) = \mathbf{0}$ since if $\deg(i) = 0$, every position in $[n]$ is dominated by a position in $[m]$ other than i . Thus $\varphi(e_i) = \varphi(\mathbf{0})$, contradicting the fact that φ is injective. \square

In view of Lemma 2, we henceforth assume w.l.o.g. that the degree of every vertex in a domination graph is at least one. In fact, we can furthermore assume that the degree of all the vertices on the right side of the bipartition is *exactly one*. Indeed, let $G = ([m] \cup [n], E)$ be a domination graph, let φ be a G -domination mapping, and suppose that a vertex $j \in [n]$ has degree $\delta > 1$. We construct G' from G by removing, arbitrarily, some $\delta - 1$ of the δ edges incident upon j . Then, obviously, φ is G' -dominating as well (cf. Example 1). It follows that any (m, n, w) -domination mapping is G -dominating for some graph $G = ([m] \cup [n], E)$ with n edges and $\deg(j) = 1$ for all $j \in [n]$. Up to a permutation of $[n]$, any such domination graph G is uniquely described by the degrees $\delta_1, \delta_2, \dots, \delta_m$ of its left vertices. We say that $(\delta_1, \delta_2, \dots, \delta_m)$ is the **degree sequence** of G , and further assume (w.l.o.g., up to a permutation of $[m]$) that $\delta_1 \leq \delta_2 \leq \dots \leq \delta_m$. With this assumption, a domination graph G can be also specified in terms of its **degree distribution** $(d_1, d_2, \dots, d_\Delta)$, where d_i is the number of left vertices of degree i and $\Delta \stackrel{\text{def}}{=} \max\{\delta_1, \delta_2, \dots, \delta_m\}$. Clearly,

$$d_1 + d_2 + \dots + d_\Delta = m \quad \text{and} \quad d_1 + 2d_2 + \dots + \Delta d_\Delta = n \quad (3)$$

A domination graph is said to be **equitable** if $\lfloor n/m \rfloor = \delta_1 \leq \delta_2 \leq \dots \leq \delta_m = \lceil n/m \rceil$. Equitable domination graphs play an important role in the study of domination mappings. In fact, we conjecture as follows.

Conjecture 1. *An (m, n, w) -domination mapping exists if and only if there exists a mapping from $\{0, 1\}^m$ to $\mathcal{B}(n, w)$ that is G -dominating for an equitable graph G .*

We now consider certain basic properties of (m, n, w) -domination mappings. The following easy lemma shows that the existence problem for such mappings is “monotonic” in both n and w .

Lemma 3. *If an (m, n, w) -domination mapping exists, then there also exist an $(m, n + 1, w)$ -domination mapping and an $(m, n, w + 1)$ -domination mapping.*

Proof. Given an (m, n, w) -domination map φ , we construct an injective map $\varphi': \{0, 1\}^m \rightarrow \mathcal{B}(n + 1, w)$ as follows: $\varphi'(x) = (\varphi(x), 0)$, where (\cdot, \cdot) stands for string concatenation. Clearly, φ' is an $(m, n + 1, w)$ -domination mapping. The other claim follows trivially from the fact that $\mathcal{B}(n, w) \subseteq \mathcal{B}(n, w + 1)$. \square

We next show that every G -domination mapping gives rise to a multitude of *derived domination mappings* corresponding to subgraphs of G induced by subsets of $[m]$.

Lemma 4. *Let $\varphi: \{0, 1\}^m \rightarrow \mathcal{B}(n, w)$ be a G -domination mapping. Fix an arbitrary left vertex i of G , and let G' be the induced subgraph of G obtained by removing i and all vertices adjacent to i . Then φ induces a G' -domination mapping φ' from $\{0, 1\}^{m-1}$ to $\mathcal{B}(n - \delta, w)$, where $\delta = \deg(i)$.*

Proof. The mapping $\varphi': \{0, 1\}^{m-1} \rightarrow \mathcal{B}(n - \delta, w)$ is derived from φ as follows. Given an arbitrary word $x' = (x_1, x_2, \dots, x_{m-1})$ in $\{0, 1\}^{m-1}$, we first extend it to $x = (x_1, x_2, \dots, x_{i-1}, 0, x_i, \dots, x_{m-1}) \in \{0, 1\}^m$. Let $\{j, j+1, \dots, j+\delta-1\} \in [n]$ be the set of positions dominated by the vertex i (where we have assumed w.l.o.g. that these positions are consecutive). Then $y = \varphi(x)$ is of the form

$$y = (y_1, y_2, \dots, y_{j-1}, \underbrace{0, 0, \dots, 0}_{\delta \text{ zeros}}, y_{j+\delta}, \dots, y_n),$$

and we set $\varphi'(x') = (y_1, y_2, \dots, y_{j-1}, y_{j+\delta}, \dots, y_n)$. It is easy to see that the mapping φ' thereby defined is an injection from $\{0, 1\}^{m-1}$ to $\mathcal{B}(n - \delta, w)$, and φ' is G' -dominating. \square

The construction of the derived mapping φ' in Lemma 4 is akin to the process of shortening linear codes in coding theory [8]. We restrict both the domain and the range of φ to subsets thereof consisting of those words that contain zeros in specified positions, then puncture out these all-zero positions to obtain φ' . Iterating this construction, a single (m, n, w) -domination mapping gives rise to 2^m derived domination mappings (some of which may be isomorphic or trivial) corresponding to all subsets of $[m]$.

An immediate consequence of Lemma 4 is that the existence problem for domination mappings is also monotonic in m . That is, for all $m \geq 2$, if an (m, n, w) -domination mapping exists, we can use Lemma 4 (in conjunction with Lemma 3) to construct an $(m - 1, n, w)$ -domination mapping. In view of this monotonicity, it is natural to define the following parameters:

$$\begin{aligned} \mu(n, w) &\stackrel{\text{def}}{=} \max\{m \in \mathbb{Z} : \text{an } (m, n, w)\text{-domination mapping exists}\} \\ \nu(m, w) &\stackrel{\text{def}}{=} \min\{n \in \mathbb{Z} : \text{an } (m, n, w)\text{-domination mapping exists}\} \end{aligned}$$

We say that an (m, n, w) -domination mapping is *optimal* if $n = \nu(m, w)$. We next discuss necessary conditions for the existence of an (m, n, w) -domination mapping, which can be regarded as bounds on $\nu(m, w)$.

3. Necessary Conditions

An (m, n, w) -domination mapping is injective by definition, so the size of its domain $\{0, 1\}^m$ cannot exceed the size of its range $\mathcal{B}(n, w)$. While trivial, this necessary condition is fundamental; we state it as follows.

Lemma 5. *For any (m, n, w) -domination mapping, we have*

$$2^m \leq \sum_{j=0}^w \binom{n}{j} \tag{4}$$

In conjunction with the shortening procedure in Lemma 4, the trivial necessary condition of Lemma 5 leads to a considerably more elaborate bound on the parameters of an (m, n, w) -domination mapping.

Lemma 6. Let $G = ([m] \cup [n], E)$ be a domination graph with degree distribution $(d_1, d_2, \dots, d_\Delta)$. Then for any G -domination mapping $\varphi: \{0, 1\}^m \rightarrow \mathcal{B}(n, w)$, we have

$$m \leq \min_{t_1, t_2, \dots, t_\Delta} \left\{ (t_1 + t_2 + \dots + t_\Delta) + \log_2 \sum_{j=0}^w \binom{n - t_1 - 2t_2 - \dots - \Delta t_\Delta}{j} \right\} \quad (5)$$

where the minimum is taken over all nonnegative integers $t_1, t_2, \dots, t_\Delta$ with $t_i \leq d_i$ for $i = 1, 2, \dots, \Delta$.

Proof. Given $t_1, t_2, \dots, t_\Delta$, we invoke Lemma 4 iteratively to obtain a domination graph G' with degree distribution $(d_1 - t_1, d_2 - t_2, \dots, d_\Delta - t_\Delta)$ and a G' -domination mapping φ' with parameters (m', n', w) , where $m' = m - (t_1 + t_2 + \dots + t_\Delta)$ and $n' = n - t_1 - 2t_2 - \dots - \Delta t_\Delta$. Applying Lemma 5 to φ' establishes (5). \square

Observe that the minimization in (5) can be reduced to the minimum of at most m terms, parametrized by $s = t_1 + t_2 + \dots + t_\Delta$ with $s \leq m - 1$. The following lemma is an important special case of Lemma 6. This lemma shows, in particular, that the trivial necessary condition of Lemma 5 is, in general, not sufficient.

Lemma 7. For any (m, n, w) -domination mapping, we have $n \geq 2m - w$.

Proof. Set $t_1 = 0$ and $t_i = d_i$ for $i = 2, 3, \dots, \Delta$ in Lemma 6. Then $n - t_1 - 2t_2 - \dots - \Delta t_\Delta = d_1$ while $t_1 + t_2 + \dots + t_\Delta = m - d_1$ in view of (3), and the bound in (5) reduces to

$$d_1 \leq \log_2 \sum_{j=0}^w \binom{d_1}{j}$$

This is only possible if $w \geq d_1$. Now, again in view of (3), we have $n \geq d_1 + 2(d_2 + \dots + d_\Delta) = 2m - d_1$, and the lemma follows. \square

The bounds on $\nu(m, w)$ resulting from Lemma 5 and Lemma 7 are illustrated in Figure 1 for $w = 3$, and we can see that neither of them implies the other. In fact, for all w , it can be readily shown that Lemma 7 is tighter than Lemma 5 whenever $w < m \leq 3w$, whereas Lemma 5 is tighter when m is large.

We next investigate the conditions under which equality can hold in the bounds of Lemma 5 and Lemma 7. A domination mapping that establishes a *bijection* between $\{0, 1\}^m$ and $\mathcal{B}(n, w)$ will be called **perfect**; clearly an (m, n, w) -domination mapping is perfect if and only if its parameters satisfy $2^m = \sum_{j=0}^w \binom{n}{j}$. There are only four known cases where (an incomplete) sum of binomial coefficients gives a power of 2. The (m, n, w) triples for these four cases are given by

$$(m, m + 1, m/2) \text{ for } m \text{ even}, \quad (m, 2^m - 1, 1), \quad (12, 90, 2), \quad (11, 23, 3) \quad (6)$$

All these cases were discovered in the context of perfect error-correcting codes [6, 11] in the 1960s, and none were found since. It is well known that perfect binary codes — that is, partitions of $\{0, 1\}^n$ into translates of the Hamming ball $\mathcal{B}(n, w)$ — exist in only three of the four cases (it is not possible to partition $\{0, 1\}^{90}$ into Hamming balls). Remarkably, we have found perfect domination mappings for each of the four triples listed in (6). The first set of parameters is attained for $m = 2$ by the $(2, 3, 1)$ -domination mapping with degree sequence $(1, 2)$, given by

$$\varphi(00) = 000, \quad \varphi(01) = 010, \quad \varphi(10) = 100, \quad \varphi(11) = 001$$

By Lemma 7, there are no $(m, m + 1, m/2)$ -domination mappings for $m \geq 4$. In Section 4.2, we explicitly construct perfect $(m, 2^m - 1, 1)$ -domination mappings for all $m \geq 1$. Finally, perfect domination mappings with parameters $(12, 90, 2)$ and $(11, 23, 3)$ were found by computer search using the methods of Section 5.

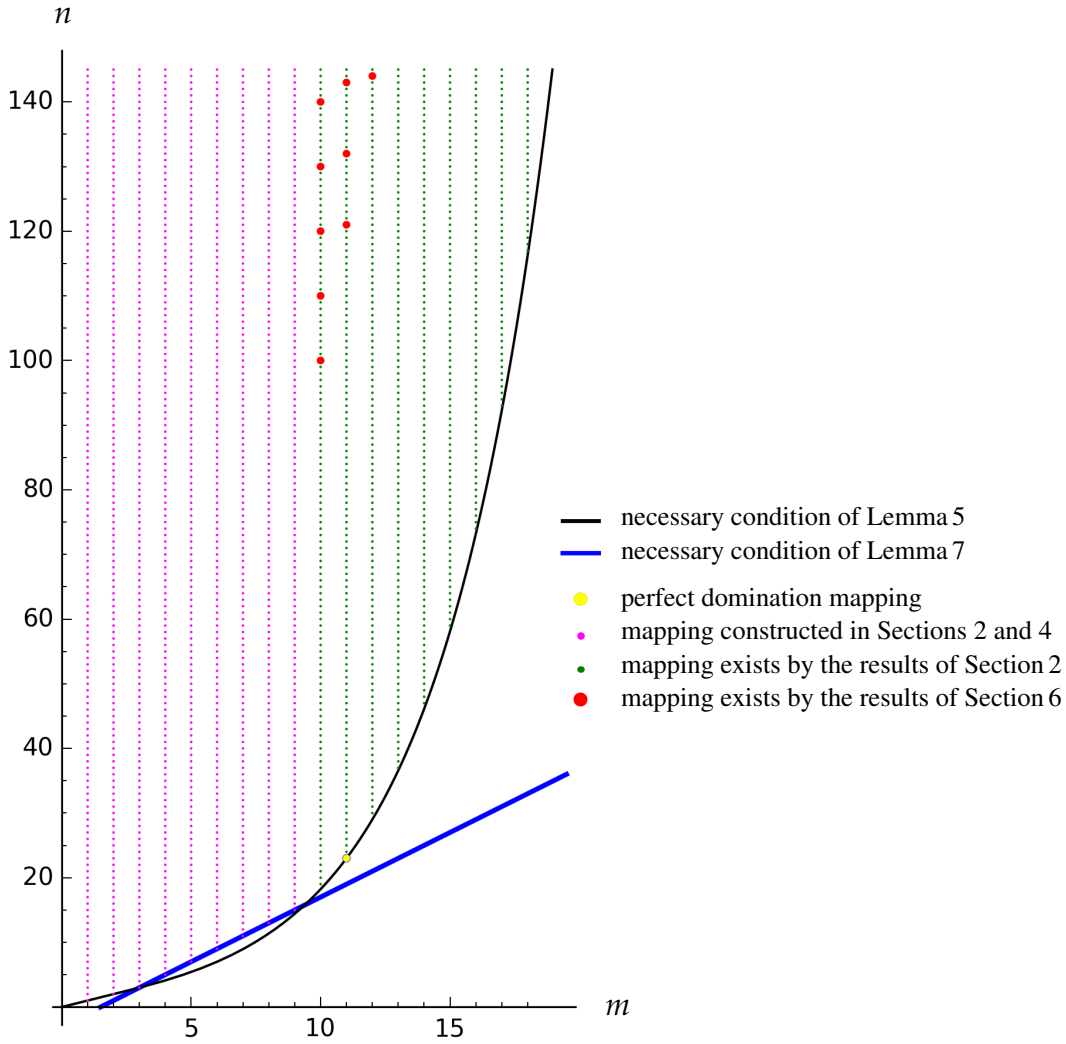


Figure 1. Bounds, constructions, and existence of (m, n, w) -domination mappings for $w = 3$.

In contrast to equality in (4), which is quite rare, equality in Lemma 7 can be achieved much more easily. The next lemma gives necessary and sufficient conditions for this.

Lemma 8. *Let φ be an (m, n, w) -domination mapping. Then $n = 2m - w$ if and only if the degree distribution of the underlying domination graph is $d_1 = w$, $d_2 = m - w$, and $d_i = 0$ for $i \geq 2$.*

Proof. It suffices to show that the conditions on the degree distribution are necessary, since the fact that they are sufficient is trivial from (3). Suppose $n = 2m - w$. Then (3) implies that

$$2m - w \geq d_1 + 2d_2 + 3(d_3 + d_4 + \cdots + d_\Delta) = d_1 + 2d_2 + 3(m - d_1 - d_2) = 3m - 2d_1 - d_2$$

which simplifies to $2d_1 + d_2 \geq m + w$. Now, it follows from (the proof of) Lemma 7 that $d_1 \leq w$ in any (m, n, w) -domination mapping. Hence $2d_1 + d_2 \geq m + w$ further implies $d_1 + d_2 \geq m$. But this is only possible if $d_1 + d_2 = m$ and $d_3 + d_4 + \cdots + d_\Delta = 0$. Finally, if $d_1 < w$, then $n \geq 2m - d_1 > 2m - w$. \square

In the next section, for all m in the range $w \leq m \leq 3w$, we present an explicit construction of domination mappings whose degree distribution satisfies the conditions of Lemma 8. By Lemma 7, such mappings are optimal, and therefore $\nu(m, w) = 2m - w$ when $w \leq m \leq 3w$ and $w \geq 3$.

Finally, we note that the case $m \leq w$ is trivial. In this case, Lemma 5 reduces to $n \geq m$. Thus $\nu(m, w) = m$ and the optimal (m, m, w) -domination mapping is the identity map from $\{0, 1\}^m$ to itself.

4. Constructions

In this section, we present several explicit *constructions* of (optimal) domination mappings. These results are distinct from our results in Sections 5 and 6, which are concerned solely with the *existence* of such mappings.

4.1. Product construction

We begin with a simple and effective recursive construction that combines any two domination mappings φ_1 and φ_2 to produce another domination mapping φ . Notably, if the mappings φ_1 and φ_2 attain the bound of Lemma 7, then so does the mapping φ obtained from this construction.

Let $\varphi_1: \{0, 1\}^{m_1} \rightarrow \mathcal{B}(n_1, w_1)$ and $\varphi_2: \{0, 1\}^{m_2} \rightarrow \mathcal{B}(n_2, w_2)$ be arbitrary domination mappings. Then their **product** $\varphi = \varphi_1 \times \varphi_2$ is a mapping from $\{0, 1\}^{m_1+m_2}$ into $\mathcal{B}(n_1 + n_2, w_1 + w_2)$ defined as follows:

$$\varphi(\mathbf{x}_1, \mathbf{x}_2) = (\varphi_1(\mathbf{x}_1), \varphi_2(\mathbf{x}_2)) \quad (7)$$

where $\mathbf{x}_1 \in \{0, 1\}^{m_1}$, $\mathbf{x}_2 \in \{0, 1\}^{m_2}$, and (\cdot, \cdot) stands for string concatenation. That is, in order to find the image of a word $\mathbf{x} \in \{0, 1\}^{m_1+m_2}$ under φ , we first parse \mathbf{x} as $(\mathbf{x}_1, \mathbf{x}_2)$, then apply φ_1 and φ_2 to the two parts.

Theorem 9. *If φ_1 is an (m_1, n_1, w_1) -domination mapping and φ_2 is an (m_2, n_2, w_2) -domination mapping, then their product $\varphi = \varphi_1 \times \varphi_2$ is an $(m_1 + m_2, n_1 + n_2, w_1 + w_2)$ -domination mapping.*

Proof. The parameters of φ and the fact that φ is injective are obvious from (7). Thus it remains to show that every position $j \in [n_1 + n_2]$ is dominated by some position $i \in [m_1 + m_2]$. But this is easy. Let G_1 and G_2 be the domination graphs for φ_1 and φ_2 , respectively. If $j \leq n_1$, we follow the corresponding edge of G_1 to find i . If $j > n_1$, we follow the edge of G_2 that corresponds to $j - n_1$ to find i^* , then set $i = i^* + m_1$. \square

Observe that the proof of Theorem 9 relates the degree distribution $(d_1, d_2, \dots, d_\Delta)$ of (the domination graph for) the product $\varphi = \varphi_1 \times \varphi_2$ to the degree distributions $(d'_1, d'_2, \dots, d'_{\Delta'})$ and $(d^*_1, d^*_2, \dots, d^*_{\Delta^*})$ of the constituent mappings φ_1 and φ_2 . Clearly $\Delta = \max\{\Delta', \Delta^*\}$ and $d_i = d'_i + d^*_i$ for all i .

Corollary 10. *For all $w \geq 3$, the product construction yields a $(3w, 5w, w)$ -domination mapping.*

Proof. Using computer search, we found domination mappings φ_3, φ_4 , and φ_5 with parameters $(9, 15, 3)$, $(12, 20, 4)$, and $(15, 25, 5)$, respectively. Since any $w \geq 3$ is an integer linear combination of 3, 4, and 5, a $(3w, 5w, w)$ -domination mapping can be constructed as the appropriate product of φ_3, φ_4 , and φ_5 . \square

Theorem 11.

$$\nu(m, w) = 2m - w \quad \text{for all } w \geq 3 \text{ and } w \leq m \leq 3w$$

Proof. In view of Lemma 7, it suffices to show that an $(m, 2m - w, w)$ -domination mapping exists for the specified ranges of m and w . We begin with the $(3w, 5w, w)$ mapping constructed in Corollary 10. Note that the degree distribution of this mapping is $(w, 2w)$ by Lemma 8. Thus for all $i = 1, 2, \dots, 2w$, we can invoke Lemma 4 to remove i vertices of degree 2, thereby producing a $(3w - i, 5w - 2i, w)$ -domination mapping. \square

We observe that optimal domination mappings that attain the bound of Lemma 7 also exist beyond the $m = 3w$ threshold. For example, there exists an optimal $(13, 22, 4)$ -domination mapping with degree distribution $(4, 9)$. Taking this mapping as both φ_1 and φ_2 in the product construction, we obtain a $(26, 44, 8)$ -domination mapping with degree distribution $(8, 18)$. This mapping is again optimal by Lemma 8.

The product construction further implies the following immediate upper bound on $\nu(m, w)$.

Lemma 12. *If $\nu(m_1, w_1) \leq n_1$ and $\nu(m_2, w_2) \leq n_2$, then $\nu(m_1 + m_2, w_1 + w_2) \leq n_1 + n_2$.*

Using Lemma 12, we can find an upper bound on $\nu(m, w)$ for all admissible parameters. We can write $m = m_1 + m_2$ and $w = w_1 + w_2$ and use the related upper bounds on $\nu(m_1, w_1)$ and $\nu(m_2, w_2)$ and apply Lemma 12. Hence, we can obtain recursively upper bounds on $\nu(m, w)$ for any triple (m, n, w) .

4.2. Domination mappings into the Hamming ball of small radius

Perfect domination mappings for $w = 1$ are easily constructed. By Lemma 5, we have $\nu(m, 1) \geq 2^m - 1$ for $w = 1$. For a given $m \geq 1$, we will construct a perfect $(m, 2^m - 1, 1)$ -domination mapping with degree sequence $(\delta_1, \delta_2, \dots, \delta_m)$, where $\delta_i = 2^i$ for all i . It should be noted that we can shorten this domination mapping to obtain a perfect $(m', 2^{m'} - 1, 1)$ -domination mapping for any $m' < m$. The $(m, 2^m - 1, 1)$ -domination mapping φ is quite simple; in fact, many such mappings exist. Clearly, $\varphi(\mathbf{0}) = \mathbf{0}$. Now, for an integer $j \in [2^m - 1]$, let $\mathbf{b}(j)$ be the binary word of length m which forms the binary representation of j . We then define $\varphi(\mathbf{b}(j)) = \mathbf{y}$, where \mathbf{y} is the binary word of length $2^m - 1$ and weight one, with the single nonzero in its j -th position. The proof of correctness for this construction can be given by induction on m or a related argument. This is summarized with the following theorem.

Theorem 13. *There exists a perfect $(m, 2^m - 1, 1)$ -domination mapping for all $m \geq 1$. Hence $\nu(m, 1) = 2^m - 1$ and $\mu(n, 1) = \lceil \log_2(n + 1) \rceil$.*

A construction of domination mappings for $w = 2$ is given in Appendix A.2. This construction attains the bound of Lemma 5 when m is odd. In fact, using these results and Section 6, we determine $\nu(m, 2)$ for all m .

5. Polynomial-Time Algorithm

In this section, we present a polynomial-time algorithm that, given any m, n , and w , determines whether an (m, n, w) -domination mapping exists for a domination graph G with an equitable degree distribution. To do so, we introduce a certain bipartite graph associated to G , which we call the *compatibility graph*.

Definition 2. *Given a domination graph $G = ([m] \cup [n], E)$, the **compatibility graph** defined by G is the bipartite graph whose vertex set is $\{0, 1\}^m \cup \mathcal{B}(n, w)$. There is an edge from $\mathbf{x} \in \{0, 1\}^m$ to $\mathbf{y} \in \mathcal{B}(n, w)$ if and only if for $(i, j) \in E$, we have that $x_i = 0$ implies $y_j = 0$.*

Fix a domination graph $G = ([m] \cup [n], E)$. For brevity, we write $\mathcal{S} \stackrel{\text{def}}{=} \{0, 1\}^m$ and $\mathcal{R} \stackrel{\text{def}}{=} \mathcal{B}(n, w)$. Also, for $\mathbf{x} \in \mathcal{S}$ and $\mathbf{y} \in \mathcal{R}$, we say that \mathbf{x} *dominates* \mathbf{y} if and only if for $(i, j) \in E$, we have that $x_i = 0$ implies $y_j = 0$. Hence, the compatibility graph defined by G has vertex set $\mathcal{S} \cup \mathcal{R}$ and $\mathbf{x} \in \mathcal{S}$ is adjacent to $\mathbf{y} \in \mathcal{R}$ if and only if \mathbf{x} dominates \mathbf{y} . The next theorem states that the existence of a G -domination mapping is equivalent to a certain graph theoretic property of the compatibility graph defined by G .

Theorem 14. *Let H be the compatibility graph defined by domination graph $G = ([m] \cup [n], E)$. There exists a G -domination mapping if and only if there exists a subgraph of H such the degree of each vertex in \mathcal{S} is exactly one and the degree of each vertex in \mathcal{R} is at most one.*

Proof. Let $\varphi : \mathcal{S} \rightarrow \mathcal{R}$ be a G -dominating mapping. We define the subgraph $H' = (\mathcal{S} \cup \mathcal{R}, E')$ where $E' = \{(\mathbf{x}, \varphi(\mathbf{x})) : \mathbf{x} \in \mathcal{S}\}$. Since \mathbf{x} dominates $\varphi(\mathbf{x})$, H' is a subgraph of H . Clearly, the degree of every vertex in \mathcal{S} is exactly one. Since φ is an injection, the degree of every vertex in \mathcal{R} is at most one.

Conversely, suppose that there exists such a subgraph $H' = (\mathcal{S} \cup \mathcal{R}, E')$. For each $\mathbf{x} \in \mathcal{S}$, define $\varphi(\mathbf{x})$ to be the unique vertex in \mathcal{R} adjacent to \mathbf{x} . Then it is readily verified that φ is a G -domination mapping. \square

Therefore, following Theorem 14, our task is reduced to determining the existence of a perfect matching in the bipartite compatibility graph defined by G . To this end, we recall the famous König-Egeváry theorem [7] for general graphs. Let $\mathcal{G} = (V, E)$ be a graph and we define the following quantities via certain integer optimization problems.

$$M(\mathcal{G}) \stackrel{\text{def}}{=} \max \left\{ \sum_{e \in E} X_e : X_e \in \{0, 1\} \text{ for all } e \in E, \sum_{e \text{ incident to } v} X_e \leq 1 \text{ for all } v \in V \right\}, \quad (8)$$

$$C(\mathcal{G}) \stackrel{\text{def}}{=} \min \left\{ \sum_{v \in V} Y_v : Y_v \in \{0, 1\} \text{ for all } v \in V, \sum_{v \text{ incident to } e} Y_v \geq 1 \text{ for all } e \in E \right\}. \quad (9)$$

Then $M(\mathcal{G})$ and $C(\mathcal{G})$ correspond to the sizes of the *maximum matching* and *minimum vertex cover*, respectively. Via weak duality, we have that $M(\mathcal{G}) \leq C(\mathcal{G})$. Next, we consider the *relaxed* or *fractional* versions of the optimization problems.

$$M^*(\mathcal{G}) \stackrel{\text{def}}{=} \max \left\{ \sum_{e \in E} X_e : 0 \leq X_e \leq 1 \text{ for all } e \in E, \sum_{e \text{ incident to } v} X_e \leq 1 \text{ for all } v \in V \right\}, \quad (10)$$

$$C^*(\mathcal{G}) \stackrel{\text{def}}{=} \min \left\{ \sum_{v \in V} Y_v : 0 \leq Y_v \leq 1 \text{ for all } v \in V, \sum_{v \text{ incident to } e} Y_v \geq 1 \text{ for all } e \in E \right\}. \quad (11)$$

We refer to $M^*(\mathcal{G})$ and $C^*(\mathcal{G})$ as the *maximum fractional matching* and *minimum fractional vertex cover*, respectively. Then we have the following inequality, $M(\mathcal{G}) \leq M^*(\mathcal{G}) \leq C^*(\mathcal{G}) \leq C(\mathcal{G})$.

For bipartite graphs, König-Egeváry theorem [7] states that $M(\mathcal{G}) = C(\mathcal{G})$. In other words, the maximum matching and the maximum fractional matching in a bipartite graph have the same size. Specifically,

$$M(\mathcal{G}) = M^*(\mathcal{G}) = C^*(\mathcal{G}) = C(\mathcal{G}).$$

5.1. Maximum matching in compatibility graphs

Recall that the vertex set in our compatibility graph is $V = \mathcal{S} \cup \mathcal{R}$ while the edge set is $E = \{(u, v) \in \mathcal{S} \times \mathcal{R} : u \text{ dominates } v\}$. Then the size of maximum matching is given by the following linear program

$$\max \left\{ \sum_{e \in E} X_e : \mathbf{A}\mathbf{X} \leq \mathbf{1}, \mathbf{X} \geq \mathbf{0} \right\}, \quad (12)$$

where \mathbf{A} is a matrix whose rows are indexed by V and columns are indexed by E . Here, $\mathbf{A}(v, e) = 1$ if v is incident to e , and $\mathbf{A}(v, e) = 0$, otherwise.

Example 1. Set $m = 2$, $n = 4$, and $w = 1$. Then $\mathcal{R} = \{00, 01, 10, 11\}$ and $\mathcal{S} = \{0000, 0001, 0010, 0100, 1000\}$. Here, $V = \mathcal{S} \cup \mathcal{R}$, and $E = \{e_i : i \in [12]\}$, where

$$\begin{aligned} e_1 &\stackrel{\text{def}}{=} (00, 0000), \\ e_2 &\stackrel{\text{def}}{=} (01, 0000), \quad e_3 \stackrel{\text{def}}{=} (01, 0001), \quad e_4 \stackrel{\text{def}}{=} (01, 0010), \\ e_5 &\stackrel{\text{def}}{=} (10, 0000), \quad e_6 \stackrel{\text{def}}{=} (10, 0100), \quad e_7 \stackrel{\text{def}}{=} (10, 1000), \\ e_8 &\stackrel{\text{def}}{=} (11, 0000), \quad e_9 \stackrel{\text{def}}{=} (11, 0001), \quad e_{10} \stackrel{\text{def}}{=} (11, 0010), \quad e_{11} \stackrel{\text{def}}{=} (11, 0100), \quad e_{12} \stackrel{\text{def}}{=} (11, 1000). \end{aligned}$$

Then the matrix A is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Set

$$\mathbf{X} = \left(1, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

We check that \mathbf{X} is indeed a feasible vector whose objective value is four. Since the size of a matching in the compatibility graph is at most four, we find that size of maximum matching in this graph is four.

Unfortunately, the number of variables in the program (12) is given by $|E| = \Theta(2^{2m-1})$ when $m \geq 2w$. Indeed, for $\mathbf{u} \in \{0, 1\}^m$, when $\text{wt}(\mathbf{u}) \geq w$, \mathbf{u} dominates all words in $\mathcal{B}(n, w)$. Since $|\mathcal{R}| = |\mathcal{B}(n, w)| \geq |\{0, 1\}^m| = 2^m$, we have that $|E| \geq (\sum_{j=w}^m \binom{m}{j})|\mathcal{R}| \geq 2^{m-1} \cdot 2^m = 2^{2m-1}$. In other words, to determine the existence of a mapping by solving this linear programme requires time *exponential* in m . In what follows, we reduce the running time to *polynomial* in m and w .

In do so, we notice that many entries in the optimal solution of Example 1 are identical. This hints that we may reduce the number of variables in the program and we do so by exploiting certain symmetries of the linear program.

5.2. Symmetries of the linear program

Let M and N be integers and let A be an $M \times N$ matrix. Consider a linear program of the form

$$\max \left\{ \sum_{i \in [N]} x_i : A\mathbf{x} \leq \mathbf{1}, \mathbf{x} \geq \mathbf{0} \right\}.$$

Let $\pi : [N] \rightarrow [N]$ be a permutation on the set of N variables. For any vector $\mathbf{x} \in \mathbb{R}^N$, let \mathbf{x}^π denote the vector whose i th component is given by $x_{\pi(i)}$. Let P_π denote the binary matrix that represents the permutation π and hence, by definition, we have that $P_\pi \mathbf{x} = \mathbf{x}^\pi$.

A permutation on the N variables can be regarded as a permutation on the N columns of A . We also consider a permutation π_{row} on the M rows of A and similarly let $P_{\pi_{\text{row}}}$ denote the binary matrix that represents the permutation π_{row} .

Definition 3. A permutation $\pi : [N] \rightarrow [N]$ is *A -preserving* if $P_{\pi_{\text{row}}} A P_\pi = A$ for some permutation $\pi_{\text{row}} : [M] \rightarrow [M]$.

This definition of A -preserving permutations can be found in linear programming literature (see Margot [9], and Bódi and Herr [1]), where the authors exploit symmetries to reduce the dimension of their linear programs. In particular, Bódi and Herr demonstrated Proposition 15 in a more general setting. Independently, Fazelli *et al.* obtained Proposition 15 in the specialized setting of finding a fractional transversal in hypergraphs. For completeness, we rederive Proposition 15 in Appendix B.

Let G_A denote a subgroup of the group of all A -preserving permutations and let G_A act on $[N]$. Suppose that the collection of orbits under this action is $\mathcal{O} = \{O_1, O_2, \dots, O_n\}$. A vector \mathbf{x} is defined to be \mathcal{O} -regular if for all $k \leq N$, $x_i = x_j$ for all $i, j \in O_k$.

Proposition 15. *Suppose that $A\mathbf{x} \leq \mathbf{1}$ and $\sum_{i=1}^N x_i = \lambda$. Then there exists an \mathcal{O} -regular vector \mathbf{x}^* such that $A\mathbf{x}^* \leq \mathbf{1}$ and $\sum_{i=1}^N x_i^* = \lambda$.*

Therefore, applying Proposition 15, we establish the following equality.

$$\max \left\{ \sum_{i \in [N]} x_i : A\mathbf{x} \leq \mathbf{1}, \mathbf{x} \geq \mathbf{0} \right\} = \max \left\{ \sum_{i \in [N]} x_i : A\mathbf{x} \leq \mathbf{1}, \mathbf{x} \geq \mathbf{0}, \mathbf{x} \text{ is } \mathcal{O}\text{-regular} \right\}.$$

In other words, we reduce the number of variables from N to n . We may then rewrite the linear program as follows:

$$\max \left\{ \sum_{i \in [n]} |O_i| x_i^* : A^* \mathbf{x}^* \leq \mathbf{1}, \mathbf{x}^* \geq \mathbf{0} \right\},$$

where A^* is an $M \times n$ matrix defined by

$$A^*(i, k) = \sum_{j \in O_k} A(i, j) \text{ for all } i \in [M], k \in [n].$$

5.3. Reducing the dimension of the linear program defined by (12)

To exploit the symmetries of our linear program, we consider a domination graph with an equitable degree distribution. Specifically, for given values of m and n , we set $\delta = \lfloor n/m \rfloor$, and

$$\begin{aligned} m_1 &= n \bmod m, & m_2 &= m - m_1, \\ n_1 &= m_1(\delta + 1), & n_2 &= m_2\delta. \end{aligned}$$

We partition $[m]$ into $I_1 \stackrel{\text{def}}{=} [m_1]$ and $I_2 \stackrel{\text{def}}{=} [m_1 + 1, m]$ (here, $[i, j]$ denotes the set $\{i, i + 1, \dots, j\}$) and assign the vertices in I_1 and I_2 the degrees $\delta + 1$ and δ , respectively. Hence, in this domination graph, there are m_1 vertices of degree $\delta + 1$ and m_2 vertices of degree δ .

We also partition $[n]$ into m groups:

$$J_i \stackrel{\text{def}}{=} \begin{cases} [(i-1)(\delta+1)+1, i(\delta+1)], & \text{if } i \in [m_1], \\ [n_1 + (i-m_1-1)\delta+1, n_1 + (i-m_1)\delta], & \text{otherwise.} \end{cases}$$

In other words, we partition $[n_1]$ into m_1 groups of size $\delta + 1$ and $[n_1 + 1, n]$ into m_2 groups of size δ .

Next, we define a group of A -preserving permutations. Here, let \mathbb{S}_X denote the set of permutations on the set X and we first produce permutations in $\mathbb{S}_{\mathcal{S}}$ and $\mathbb{S}_{\mathcal{R}}$. Consider a permutation $\gamma \in \mathbb{S}_{I_1} \times \mathbb{S}_{I_2}$. Since I_1 and I_2 partition $[m]$, the permutation γ belongs to $\mathbb{S}_{[m]}$ and we define $\pi_{\gamma}^{\mathcal{S}} \in \mathbb{S}_{\mathcal{S}}$ such that $\pi_{\gamma}^{\mathcal{S}}(\mathbf{u}) = \mathbf{u}^{\gamma}$ for $\mathbf{u} \in \mathcal{S}$. We then consider the subset

$$\mathbb{B} = \{\beta \in \mathbb{S}_{\mathcal{R}} : \beta(v)|_{J_i} \neq \mathbf{0} \text{ if and only if } v|_{J_i} \neq \mathbf{0} \text{ for all } i \in [m], v \in \mathcal{S}\}.$$

It can be verified that \mathbb{B} is a subgroup of $\mathbb{S}_{\mathcal{R}}$. Given $\gamma \in \mathbb{S}_{I_1} \times \mathbb{S}_{I_2}$ and $\beta \in \mathbb{B}$, we define $\pi_{\gamma, \beta}^{\mathcal{R}} \in \mathbb{S}_{\mathcal{R}}$ to be the permutation such that $\pi_{\gamma, \beta}^{\mathcal{R}}(v)$ is the word obtained rearranging the m subwords in $\beta(v)$ in accordance to γ . Finally, we obtain a permutation $\pi_{\gamma, \beta}$ on $\mathcal{S} \times \mathcal{R}$ by simply setting $\pi_{\gamma, \beta}(\mathbf{u}, v) = (\pi_{\gamma}^{\mathcal{S}}(\mathbf{u}), \pi_{\gamma, \beta}^{\mathcal{R}}(v))$. Set $\mathbb{G} = \{\pi_{\gamma, \beta} : \gamma \in \mathbb{S}_{I_1} \times \mathbb{S}_{I_2}, \beta \in \mathbb{B}\}$. Abusing notation, we simply write $\mathbb{G} = (\mathbb{S}_{I_1} \times \mathbb{S}_{I_2}) \times \mathbb{B}$.

Example 2.

- (i) (Example 1 continued.) Consider $m = 2$, $n = 4$, and $w = 1$. Then $\delta = 2$, $m_1 = n_1 = 0$, $m_2 = 2$ and $n_2 = 4$.

Label the words in \mathcal{R} as follow: $v_0 = 0000$, $v_1 = 0001$, $v_2 = 0010$, $v_3 = 0100$, and $v_4 = 1000$. So, $\mathbb{B} = \{(v_0, v_1, v_2, v_3, v_4), (v_0, v_2, v_1, v_3, v_4), (v_0, v_1, v_2, v_4, v_3), (v_0, v_2, v_1, v_4, v_3)\}$. Since $m_1 = 0$, $\mathbb{G} = \mathbb{S}_{[2]} \times \mathbb{B}$.

Let $\gamma = (2, 1)$ and $\beta = (v_0, v_2, v_1, v_3, v_4)$. Then $\pi_\gamma^{\mathcal{S}}(01) = 10$, $\pi_{\gamma, \beta}^{\mathcal{R}}(0001) = 1000$ and so, $\pi_{\gamma, \beta}(01, 0001) = (10, 1000)$. Consider the orbit of $(01, 0001)$ which is $\{\pi(01, 0001) : \pi \in \mathbb{G}\}$. The orbit is given by

$$\{(01, 0001), (01, 0010), (10, 0100), (10, 1000)\}.$$

- (ii) Consider $m = 3$, $n = 4$, and $w = 2$. Then $\delta = 1$, $m_1 = 1$, $m_2 = 2$, $n_1 = 2$ and $n_2 = 2$.

The orbit of $(101, 0101)$ is given by $\{(101, 0101), (101, 1001), (110, 0110), (110, 1010)\}$, while the orbit of $(100, 0100)$ is given by $\{(100, 0100), (100, 1000)\}$.

Recall that the columns of A are indexed by $E \subseteq \mathcal{S} \times \mathcal{R}$, while \mathbb{G} is a subgroup of $\mathbb{S}_{\mathcal{S} \times \mathcal{R}}$. In the next lemma, we show that \mathbb{G} may be regarded as a subgroup of \mathbb{S}_E . We do this by showing that the image of E under the permutation $\pi_{\gamma, \beta}$ remains as E .

Lemma 16. *Let $\pi_{\gamma, \beta} \in \mathbb{G}$, $\mathbf{u} \in \mathcal{S}$ and $\mathbf{v} \in \mathcal{R}$. Then \mathbf{u} dominates \mathbf{v} if and only if $\pi_\gamma^{\mathcal{S}}(\mathbf{u})$ dominates $\pi_{\gamma, \beta}^{\mathcal{R}}(\mathbf{v})$.*

Proof. Since \mathbb{G} is a group, it suffices to prove in one direction. Let $\text{supp}_\delta(\mathbf{v}) \stackrel{\text{def}}{=} \{i \in [m] : \mathbf{v}|_{j_i} \neq \mathbf{0}\}$. Since \mathbf{u} dominates \mathbf{v} , we have that $\text{supp}(\mathbf{u}) \supseteq \text{supp}_\delta(\mathbf{v})$.

Since $\beta \in \mathbb{B}$, we have that $\text{supp}_\delta(\mathbf{v}) = \text{supp}_\delta(\beta(\mathbf{v}))$. From the definition of $\pi_\gamma^{\mathcal{S}}$ and $\pi_{\gamma, \beta}^{\mathcal{R}}$, we have that $\text{supp}(\pi_\gamma^{\mathcal{S}}(\mathbf{u})) = \gamma(\text{supp}(\mathbf{u}))$ and $\text{supp}_\delta(\pi_{\gamma, \beta}^{\mathcal{R}}(\mathbf{v})) = \gamma(\text{supp}_\delta(\beta(\mathbf{v}))) = \gamma(\text{supp}_\delta(\mathbf{v}))$. Therefore, $\text{supp}(\pi_\gamma^{\mathcal{S}}(\mathbf{u})) \supseteq \text{supp}_\delta(\pi_{\gamma, \beta}^{\mathcal{R}}(\mathbf{v}))$ and so, $\pi_\gamma^{\mathcal{S}}(\mathbf{u})$ dominates $\pi_{\gamma, \beta}^{\mathcal{R}}(\mathbf{v})$. \square

Therefore, since $E = \{(\mathbf{u}, \mathbf{v}) \in \mathcal{S} \times \mathcal{R} : \mathbf{u} \text{ dominates } \mathbf{v}\}$, we have that \mathbb{G} is a subgroup of \mathbb{S}_E .

Lemma 17. *Every permutation in \mathbb{G} is a A -preserving permutation.*

Proof. Let $\pi = \pi_{\gamma, \beta} \in \mathbb{G}$. To show that π is A -preserving, it suffices to provide a permutation π_{row} on the rows of A such that $P_{\pi_{\text{row}}} A P_\pi = A$, or

$$A P_\pi = P_{\pi_{\text{row}}}^{-1} A = P_{\pi_{\text{row}}^{-1}} A. \quad (13)$$

Let $(\mathbf{u}, \mathbf{v}) \in E$ and $\mathbf{z} \in \mathcal{S} \cup \mathcal{R}$. Let the entry of A at row \mathbf{z} and column (\mathbf{u}, \mathbf{v}) , or the $(\mathbf{z}, (\mathbf{u}, \mathbf{v}))$ th entry of A , be written as $A(\mathbf{z}, (\mathbf{u}, \mathbf{v}))$. Then the $(\mathbf{z}, (\mathbf{u}, \mathbf{v}))$ th entry of $A P_\pi$ is given by $A(\mathbf{z}, \pi^{-1}(\mathbf{u}, \mathbf{v}))$.

We next consider the permutations on the rows. Note that $\pi_\gamma^{\mathcal{S}} \in \mathbb{S}_{\mathcal{S}}$ and $\pi_{\gamma, \beta}^{\mathcal{R}} \in \mathbb{S}_{\mathcal{R}}$. So, let \mathbb{G} act on $\mathcal{S} \cup \mathcal{R}$ by setting $\pi(\mathbf{z}) = \pi_\gamma^{\mathcal{S}}(\mathbf{z})$ when $\mathbf{z} \in \mathcal{S}$ and $\pi(\mathbf{z}) = \pi_{\gamma, \beta}^{\mathcal{R}}(\mathbf{z})$ when $\mathbf{z} \in \mathcal{R}$. Set $\pi_{\text{row}} = \pi^{-1}$, the inverse of π in \mathbb{G} . Then the $(\mathbf{z}, (\mathbf{u}, \mathbf{v}))$ th entry of $P_{\pi_{\text{row}}}^{-1} A$ is given by $A(\pi_{\text{row}}^{-1}(\mathbf{z}), (\mathbf{u}, \mathbf{v})) = A(\pi(\mathbf{z}), (\mathbf{u}, \mathbf{v}))$.

Finally, to establish (13) we show that $A(\mathbf{z}, \pi^{-1}(\mathbf{u}, \mathbf{v})) = A(\pi(\mathbf{z}), (\mathbf{u}, \mathbf{v}))$. This then follows from the fact that

$$A(\mathbf{z}, (\mathbf{u}, \mathbf{v})) = \begin{cases} 1, & \text{if } \mathbf{z} = \mathbf{u} \text{ or } \mathbf{z} = \mathbf{v}, \\ 0, & \text{otherwise.} \end{cases}$$

\square

We next study the orbits of $\mathcal{S} \times \mathcal{R}$ under this group action of \mathbb{G} . To this end, we define the (m_1, m_2) -weights and the $(n_1, n_2; \delta)$ -weights of words in \mathcal{S} and \mathcal{R} , respectively. For $\mathbf{u} \in \mathcal{S}$ and $\mathbf{v} \in \mathcal{R}$, define

$$\begin{aligned} (m_1, m_2)\text{-wt}(\mathbf{u}) &= (\sigma_1, \sigma_2), \text{ where } \sigma_i = \text{wt}(\mathbf{u}|_{I_i}) \text{ for } i \in [2], \\ (n_1, n_2; \delta)\text{-wt}(\mathbf{v}) &= (\rho_1, \rho_2), \text{ where } \rho_i = |\{j \in I_j : \mathbf{v}|_{I_j} \neq \mathbf{0}\}| \text{ for } i \in [2]. \end{aligned}$$

Using these weights, we then characterise the orbits of $\mathcal{S} \times \mathcal{R}$ under this group action of \mathbb{G} .

Lemma 18. *Let $(\mathbf{u}, \mathbf{v}), (\mathbf{u}', \mathbf{v}') \in E$. (\mathbf{u}, \mathbf{v}) and $(\mathbf{u}', \mathbf{v}')$ belong to the same orbit if and only if $(m_1, m_2)\text{-wt}(\mathbf{u}) = (m_1, m_2)\text{-wt}(\mathbf{u}')$ and $(n_1, n_2; \delta)\text{-wt}(\mathbf{v}) = (n_1, n_2; \delta)\text{-wt}(\mathbf{v}')$.*

Proof. If (\mathbf{u}, \mathbf{v}) and $(\mathbf{u}', \mathbf{v}')$ belong to the same orbit, then $\pi_{\gamma, \beta}(\mathbf{u}, \mathbf{v}) = (\mathbf{u}', \mathbf{v}')$ for some $\pi_{\gamma, \beta} \in \mathbb{G}$. In other words, $\pi_{\gamma}^{\mathcal{S}}(\mathbf{u}) = \mathbf{u}'$ and $\pi_{\gamma, \beta}^{\mathcal{R}}(\mathbf{v}) = \mathbf{v}'$. Since $\gamma \in \mathbb{S}_{I_1} \times \mathbb{S}_{I_2}$, we have $\text{wt}(\mathbf{u}|_{I_i}) = \text{wt}(\mathbf{u}'|_{I_i})$ for $i \in [2]$, and so, $(m_1, m_2)\text{-wt}(\mathbf{u}) = (m_1, m_2)\text{-wt}(\mathbf{u}')$. Similarly, since $\beta \in \mathbb{B}$, we have $|\{j \in I_j : \mathbf{v}|_{I_j} \neq \mathbf{0}\}| = |\{j \in I_j : \mathbf{v}'|_{I_j} \neq \mathbf{0}\}|$ and hence, $(n_1, n_2; \delta)\text{-wt}(\mathbf{v}) = (n_1, n_2; \delta)\text{-wt}(\mathbf{v}')$.

Conversely, suppose that $(m_1, m_2)\text{-wt}(\mathbf{u}) = (m_1, m_2)\text{-wt}(\mathbf{u}') = (\sigma_1, \sigma_2)$ and $(n_1, n_2; \delta)\text{-wt}(\mathbf{v}) = (n_1, n_2; \delta)\text{-wt}(\mathbf{v}') = (\rho_1, \rho_2)$. Consider the words

$$\begin{aligned} \mathbf{u}^* &= \overbrace{00 \cdots 011 \cdots 1}^{m_1} \overbrace{00 \cdots 011 \cdots 1}^{m_2}, \\ &\quad \sigma_1 \qquad \qquad \qquad \sigma_2 \\ \mathbf{v}^* &= \overbrace{0 \cdots 0}^{\delta+1} \cdots \overbrace{0 \cdots 0}^{\delta+1} \overbrace{0 \cdots 01}^{\delta+1} \cdots \overbrace{0 \cdots 01}^{\delta+1} \overbrace{0 \cdots 0}^{\delta} \cdots \overbrace{0 \cdots 0}^{\delta} \overbrace{0 \cdots 01}^{\delta} \cdots \overbrace{0 \cdots 01}^{\delta}. \\ &\quad m_1 - \rho_1 \qquad \qquad \rho_1 \qquad \qquad m_2 - \rho_2 \qquad \qquad \rho_2 \end{aligned}$$

Then $(m_1, m_2)\text{-wt}(\mathbf{u}^*) = (\sigma_1, \sigma_2)$ and $(n_1, n_2; \delta)\text{-wt}(\mathbf{v}^*) = (\rho_1, \rho_2)$. We find a permutation in \mathbb{G} that maps $(\mathbf{u}^*, \mathbf{v}^*)$ to (\mathbf{u}, \mathbf{v}) . Let γ^{-1} be a permutation in $\gamma \in \mathbb{S}_{I_1} \times \mathbb{S}_{I_1}$ that rearranges the coordinates the m coordinates of \mathbf{u} and the m subwords of \mathbf{v} such that $(\mathbf{u}^{\gamma^{-1}}, \hat{\mathbf{v}})$ is the lexicographical smallest amongst all permutations. Let $\beta \in \mathbb{B}$ be a permutation that maps \mathbf{v}^* to $\hat{\mathbf{v}}$. Then $\pi_{\gamma, \beta}(\mathbf{u}^*, \mathbf{v}^*) = (\mathbf{u}, \mathbf{v})$. Since \mathbb{G} is a subgroup of \mathbb{S}_E , we can then find a permutation that maps (\mathbf{u}, \mathbf{v}) to $(\mathbf{u}', \mathbf{v}')$. \square

Following Lemma 18, we can then index each orbit with the quadruple $(\sigma_1, \sigma_2, \rho_1, \rho_2)$, where $(\sigma_1, \sigma_2) = (m_1, m_2)\text{-wt}(\mathbf{u})$ and $(\rho_1, \rho_2) = (n_1, n_2; \delta)\text{-wt}(\mathbf{v})$ for some (\mathbf{u}, \mathbf{v}) in the orbit. In particular, the index set is given by

$$\Omega \stackrel{\text{def}}{=} \{(\sigma_1, \sigma_2, \rho_1, \rho_2) : 0 \leq \sigma_i \leq m_i \text{ for } i \in [2], 0 \leq \rho_1 \leq \min\{\sigma_1, w\}, 0 \leq \rho_2 \leq \min\{\sigma_2, w - \rho_1\}\}.$$

Hence, the number of variables is reduced to $O(m^2 w^2)$. Besides reducing the number of variables, the group action also identifies certain constraints. In particular, the $2^m + \sum_{j=0}^w \binom{n}{j}$ constraints are reduced to $O(m^2 + w^2)$ constraints. Then by the equivalence of the linear programs, we are able to determine the existence of the desired mapping in time polynomial in m and w .

Next, we compute the size of the orbits and then state the reduced linear program. Let $0 \leq \rho_1, \rho_2 \leq w$ and $0 \leq \rho_1 + \rho_2 \leq w$ and define the quantity

$$C_{\rho_1 \rho_2} \stackrel{\text{def}}{=} \sum_{\substack{w_1^{(1)} + \cdots + w_{\rho_1}^{(1)} + w_1^{(2)} + \cdots + w_{\rho_2}^{(2)} \leq w \\ w_j^{(i)} \geq 1}} \prod_{j=1}^{\rho_1} \binom{\delta+1}{w_j^{(1)}} \prod_{j=1}^{\rho_2} \binom{\delta}{w_j^{(2)}}.$$

Here, $C_{\rho_1 \rho_2}$ computes the number of words \mathbf{v} in \mathcal{R} such that $\{j \in I_1 : \mathbf{v}|_{I_j} \neq \mathbf{0}\} = L_1$ and $\{j \in I_2 : \mathbf{v}|_{I_j} \neq \mathbf{0}\} = L_2$ for some ρ_1 -subset L_1 of I_1 and ρ_2 -subset L_2 of I_2 .

Lemma 19. Fix $(\sigma_1, \sigma_2, \rho_1, \rho_2) \in \Omega$.

(i) The number of pairs (\mathbf{u}, \mathbf{v}) in E with (m_1, m_2) -wt $(\mathbf{u}) = (\sigma_1, \sigma_2)$ and $(n_1, n_2; \delta)$ -wt $(\mathbf{v}) = (\rho_1, \rho_2)$ is $\binom{m_1}{\sigma_1} \binom{m_2}{\sigma_2} \binom{\sigma_1}{\rho_1} \binom{\sigma_2}{\rho_2} C_{\rho_1 \rho_2}$.

(ii) Fix $\mathbf{u} \in \mathcal{S}$ with (m_1, m_2) -wt $(\mathbf{u}) = (\sigma_1, \sigma_2)$. The number of words $\mathbf{v} \in \mathcal{R}$ that are dominated by \mathbf{u} with $(n_1, n_2; \delta)$ -wt $(\mathbf{v}) = (\rho_1, \rho_2)$ is $\binom{\sigma_1}{\rho_1} \binom{\sigma_2}{\rho_2} C_{\rho_1 \rho_2}$.

(iii) Fix $\mathbf{v} \in \mathcal{R}$ with $(n_1, n_2; \delta)$ -wt $(\mathbf{v}) = (\sigma_1, \sigma_2)$. The number of words $\mathbf{u} \in \mathcal{S}$ that dominates \mathbf{v} with (m_1, m_2) -wt $(\mathbf{u}) = (\sigma_1, \sigma_2)$ is $\binom{m_1 - \rho_1}{\sigma_1 - \rho_1} \binom{m_2 - \rho_2}{\sigma_2 - \rho_2}$.

Proof. For $(\mathbf{u}, \mathbf{v}) \in E$, we set K_i to be the support of $\mathbf{u}|_{I_i}$ and $L_i = \{j \in I_i : \mathbf{v}|_{J_j} \neq \mathbf{0}\}$ for $i \in [2]$. Note that if (m_1, m_2) -wt $(\mathbf{u}) = (\sigma_1, \sigma_2)$ and $(n_1, n_2; \delta)$ -wt $(\mathbf{v}) = (\rho_1, \rho_2)$, then $|K_i| = \sigma_i$ and $|L_i| = \rho_i$ for $i \in [2]$.

(i) There are $\binom{m_1}{\sigma_1} \binom{m_2}{\sigma_2}$ to choose words \mathbf{u} with (m_1, m_2) -wt $(\mathbf{u}) = (\sigma_1, \sigma_2)$. Since \mathbf{u} dominates \mathbf{v} , we have $\binom{\sigma_1}{\rho_1} \binom{\sigma_2}{\rho_2}$ choices for L_1 and L_2 . For fixed L_1 and L_2 , there are $C_{\rho_1 \rho_2}$ choices for \mathbf{v} . Therefore, the total number of pairs is $\binom{s_1}{\sigma_1} \binom{s_2}{\sigma_2} \binom{\sigma_1}{\rho_1} \binom{\sigma_2}{\rho_2} C_{\rho_1 \rho_2}$.

(ii) When K_1 and K_2 are fixed, the previous argument demonstrates that there are $\binom{\sigma_1}{\rho_1} \binom{\sigma_2}{\rho_2} C_{\rho_1 \rho_2}$ choices for \mathbf{v} .

(iii) When L_1 and L_2 are fixed, there are $\binom{m_1 - \rho_1}{\sigma_1 - \rho_1}$ and $\binom{m_2 - \rho_2}{\sigma_2 - \rho_2}$ choices for K_1 and K_2 , respectively. Therefore, the desired number is $\binom{s_1 - \rho_1}{\sigma_1 - \rho_1} \binom{s_2 - \rho_2}{\sigma_2 - \rho_2}$.

□

Finally, we state the reduced linear program.

$$\max \sum_{(\sigma_1, \sigma_2, \rho_1, \rho_2) \in \Omega} \binom{m_1}{\sigma_1} \binom{m_2}{\sigma_2} \binom{\sigma_1}{\rho_1} \binom{\sigma_2}{\rho_2} C_{\rho_1 \rho_2} X_{\sigma_1 \sigma_2 \rho_1 \rho_2} \quad (14)$$

subject to the following constraints.

(I) \mathcal{S} -side constraints:

$$\sum_{\rho_1=0}^{\min\{\sigma_1, w\}} \sum_{\rho_2=0}^{\min\{\sigma_2, w - \rho_1\}} \binom{\sigma_1}{\rho_1} \binom{\sigma_2}{\rho_2} C_{\rho_1 \rho_2} X_{\sigma_1 \sigma_2 \rho_1 \rho_2} \leq 1 \text{ for all } 0 \leq \sigma_1 \leq m_1, 0 \leq \sigma_2 \leq m_2.$$

(II) \mathcal{R} -side constraints:

$$\sum_{\sigma_1=\rho_1}^{m_1} \sum_{\sigma_2=\rho_2}^{m_2} \binom{m_1 - \rho_1}{\sigma_1 - \rho_1} \binom{m_2 - \rho_2}{\sigma_2 - \rho_2} X_{\sigma_1 \sigma_2 \rho_1 \rho_2} \leq 1 \text{ for all } 0 \leq \rho_1 \leq m_1, 0 \leq \rho_2 \leq m_2, 0 \leq \rho_1 + \rho_2 \leq w.$$

(III) Variable constraints:

$$0 \leq X_{\sigma_1 \sigma_2 \rho_1 \rho_2} \leq 1 \text{ for all } (\sigma_1, \sigma_2, \rho_1, \rho_2) \in \Omega.$$

There exists a mapping if and only if the objective value achieves 2^m . Furthermore, we know that the objective value attains 2^m if and only if the \mathcal{S} -side constraints are active, or met with equality.

Example 3.

- (i) (Example 1 continued.) Consider $m = 2$, $n = 4$, and $w = 1$. Then $\delta = 2$, $m_1 = n_1 = 0$, $m_2 = 2$, $n_2 = 4$.

Hence, E is partitioned into the orbits

$$\mathcal{O} = \{O_{(0,0)} = \{e_1\}, O_{(1,0)} = \{e_2, e_5\}, O_{(1,1)} = \{e_3, e_4, e_6, e_7\}, O_{(2,0)} = \{e_8\}, O_{(2,1)} = \{e_9, e_{10}, e_{11}, e_{12}\}\},$$

with

$$\Omega = \{(0,0), (1,0), (1,1), (2,0), (2,1)\}.$$

Then A^* is given by the 5×5 matrix

$$A^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

- (ii) Consider $m = 3$, $n = 4$, and $w = 2$. Then $\delta = 1$, $m_1 = 1$, $m_2 = 2$, $n_1 = 2$ and $n_2 = 2$. Here,

$$\Omega = \{(0,0,0,0), (0,1,0,1), (0,1,0,1), (0,2,0,0), (0,2,0,1), (0,2,0,2), (1,0,0,0), (1,0,1,0), (1,1,0,0), (1,1,0,1), (1,1,1,0), (1,1,1,1), (1,2,0,0), (1,2,0,1), (1,2,0,2), (1,2,1,0), (1,2,1,1)\}.$$

Then A^* is given by the 11×17 matrix

$$A^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 3 & 4 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We summarize the results in this section with the following theorem.

Theorem 20. *Let $G = ([m] \cup [n], E)$ be an equitable domination graph. The linear program defined by (14) is equivalent to (12) and has $O(m^2 w^2)$ variables and $O(m^2 + w^2)$ constraints. Therefore, we can determine the existence of an G -domination mapping in time polynomial in m and w .*

6. Existence Proof

In this section, we prove Theorem 1. In particular, we show that for sufficiently large m , an (m, n, w) -domination mapping exists whenever $|\mathcal{B}(n, w)| \geq 2^m$. From Theorem 14, this is equivalent to finding a perfect matching in the associated compatibility graph. To this end, we invoke the celebrated Hall's marriage theorem [5].

Theorem 21. Consider a bipartite graph with V_1 and V_2 as its left and right vertices, and $|V_1| \leq |V_2|$. There is a matching of size $|V_1|$ (i.e. a perfect matching) if and only if for each subset X of V_1 , the number of vertices in the neighborhood of X has at least $|X|$ vertices.

In what follows, we study the sufficiency condition of Theorem 21 for our compatibility graph. In particular, a set of vertices $X \subset \mathcal{S}$ is called a *bad set* if its neighbourhood $N(X)$ in \mathcal{R} has size less than $|X|$. In other words, $|X| > |N(X)|$. We examine properties of bad sets and find compatibility graphs that contain no bad sets. Therefore, in such compatibility graphs, the condition of Theorem 21 is satisfied and hence the corresponding (m, n, w) -domination mapping exists.

For our exposition, we define certain quantities and properties for sets of vertices in \mathcal{S} . First, we define the notion of descendant-closed.

Definition 4. Let $u, v \in \mathcal{S}$. Recall that $v \prec u$ if $\text{supp}(v) \subseteq \text{supp}(u)$ and call v a descendant of u . A set X is *descendant-closed* or *d-closed* if for all $u \in X$, $v \prec u$ implies that $v \in X$.

Given any bad set, we are always able to construct a (possibly different) d-closed bad set with the same cardinality.

Lemma 22 (Closure Lemma). *If X is a bad set, then there exists a d-closed bad set X' for which $|X| = |X'|$.*

Proof. Suppose X is not d-closed. Then there exists $u \in X$ and $v \in \mathcal{S}$ such that $v \prec u$ but $v \notin X$. Set $X^{(1)} = X \cup \{v\} \setminus \{u\}$. Then $|X^{(1)}| = |X|$ and $|N(X^{(1)})| \leq |N(X)| < |X| = |X^{(1)}|$. Hence, $X^{(1)}$ is a bad set with the same size as X . If $X^{(1)}$ is not d-closed, then we similarly construct $X^{(2)}$ such that $|X^{(2)}| = |X|$ and $X^{(2)}$ is bad. Since this process is finite, we eventually obtain a d-closed set $X^{(m)}$ such that $|X^{(m)}| = |X|$ and $X^{(m)}$ is bad. \square

Next, we define the notion of balanced sets.

Definition 5. Let $i \in [m]$. We say that X is *i-balanced* if the following holds. If $ab \in X$ with $|a| = i$, then $a'b \in X$ for all $a' \in \{0, 1\}^i$.

Observe that if X is i -balanced, then X is j -balanced for all $j \leq i$. As a convention, we say that all sets X are 0-balanced. Finally, we look at certain collection of bad sets that are both d-closed and balanced.

Definition 6. Let $0 \leq i \leq s$. Let \mathcal{X}_i be the collection of all bad sets that are d-closed and i -balanced.

Our proof on the non-existence of bad sets follows an induction strategy.

- (I) Assume that there exists a bad set X_0 .
- (II) **Base case.** Using the closure lemma, we may modify X_0 so that X'_0 is both d-closed and 0-balanced. This implies that \mathcal{X}_0 is nonempty.
- (III) **Induction step.** Let $1 \leq i \leq m$. By induction hypothesis, we have that \mathcal{X}_{i-1} is nonempty. Pick $X \in \mathcal{X}_{i-1}$ with the smallest cardinality. We then construct a set X' such that $X' \in \mathcal{X}_i$. This implies that \mathcal{X}_i is nonempty.
- (IV) Therefore, \mathcal{X}_m is nonempty and $X \in \mathcal{X}_m$ implies that $X = \mathcal{S}$. This contradicts the fact that $|\mathcal{S}| \leq |\mathcal{R}| = |N(\mathcal{S})|$.

In what follows, we focus on the induction step. We fix $X \in \mathcal{X}_{i-1}$ and augment X so that the resulting X' is in \mathcal{X}_i . To ensure that X' is bad, we need to compute the increase in the size of the neighbourhood. Formally, we have the following definition.

Definition 7. Let $U, V \subseteq \mathcal{S}$. Set $\Xi_U(V) \stackrel{\text{def}}{=} |N(U \cup V)| - |N(U)|$. In other words, $\Xi_U(V)$ is the number of additional neighbours that V adds when adjoined to U .

When X is the smallest set in \mathcal{X}_{i-1} , we have the following property of $\Xi_{X \setminus Y}(Y)$ for certain subset $Y \subseteq X$.

Lemma 23 (Removal Lemma). Let X be the smallest set in \mathcal{X}_i . Let $Y \subseteq X$ such that $X \setminus Y$ is both d -closed and i -balanced. Then $|Y| > \Xi_{X \setminus Y}(Y)$.

Proof. Since X is a bad set, we have that $|X| > |N(X)|$. By the minimality of X , we have that $X \setminus Y$ is not a bad set, because $X \setminus Y$ is both d -closed and i -balanced and $|X \setminus Y| < |X|$. Therefore, $|X \setminus Y| \leq |N(X \setminus Y)|$. Hence, $|Y| = |X| - |X \setminus Y| > |N(X)| - |N(X \setminus Y)| = \Xi_{X \setminus Y}(Y)$. \square

Next, we describe how we augment X . Define

$$Y = \{\mathbf{u} \in X : u_i = 0, \mathbf{u} + \mathbf{e}_i \notin X\}, \text{ and}$$

$$Z = Y + \mathbf{e}_i.$$

We have the following lemmas on $X \cup Z$ and $X \setminus Y$.

Lemma 24. If X is d -closed and $(i-1)$ -balanced, then $X \cup Z$ is both d -closed and i -balanced.

Proof. First, we show that $X \cup Z$ is d -closed. Let $\mathbf{u} \in X \cup Z$ and $\mathbf{v} \prec \mathbf{u}$. If $\mathbf{u} \in X$, then $\mathbf{v} \in X \subseteq X \cup Z$ since X is d -closed. Otherwise, $\mathbf{u} \in Z$ and hence, $\mathbf{u} = \mathbf{u}' + \mathbf{e}_i$ with $\mathbf{u}' \in X$. If $v_i = 0$, then $\mathbf{v} \prec \mathbf{u}'$ and so, $\mathbf{v} \in X$ since X is d -closed. Hence, $\mathbf{v} \in X \cup Z$. If $v_i = 1$, then $\mathbf{v} - \mathbf{e}_i \prec \mathbf{u}'$ and so, $\mathbf{v} - \mathbf{e}_i \in X$. Hence, $\mathbf{v} = (\mathbf{v} - \mathbf{e}_i) + \mathbf{e}_i \in Z \subseteq X \cup Z$. Thus, $X \cup Z$ is d -closed.

Next, we show that $X \cup Z$ is i -balanced. Let $\mathbf{a}\mathbf{b} \in X \cup Z$ with $|\mathbf{a}| = i$ and let $\mathbf{c} \in \{0, 1\}^i$. Let \mathbf{a}' and \mathbf{c}' be the first $i-1$ symbols of \mathbf{a} and \mathbf{c} , respectively. So, $\mathbf{a} = \mathbf{a}'a_i$ and $\mathbf{c} = \mathbf{c}'c_i$. Since $X \cup Z$ is d -closed, it suffices to consider the case, $c_i = 1$.

In other words, we want to show that $\mathbf{c}'1\mathbf{b} \in X \cup Z$. We distinguish between two cases:

- (a) $\mathbf{a}'1\mathbf{b} \in X$. Since X is $(i-1)$ -balanced, we have $\mathbf{c}'1\mathbf{b} \in X \subseteq X \cup Z$.
- (b) $\mathbf{a}'1\mathbf{b} \notin X$. This implies that $\mathbf{a}'0\mathbf{b} \in X$ by the definition of Y and Z . Again, since X is $(i-1)$ -balanced, we have $\mathbf{c}'0\mathbf{b} \in X$ and so, $\mathbf{c}'1\mathbf{b} = \mathbf{c}'0\mathbf{b} + \mathbf{e}_i \in X \cup Z$.

Thus, $X \cup Z$ is i -balanced. \square

Lemma 25. If X is d -closed and $(i-1)$ -balanced, then $X \setminus Y$ is both d -closed and $(i-1)$ -balanced.

Proof. First, we show that $X \setminus Y$ is d -closed. Let $\mathbf{u} \in X \setminus Y$ and $\mathbf{v} \prec \mathbf{u}$. Since $\mathbf{u} \notin Y$, either $u_i = 1$ or $\mathbf{u} + \mathbf{e}_i \in X$. Hence, set $\mathbf{u}' = \mathbf{u}$ if $u_i = 1$ and $\mathbf{u}' = \mathbf{u} + \mathbf{e}_i$ otherwise. If $v_i = 0$, then $\mathbf{v} + \mathbf{e}_i \prec \mathbf{u}' \in X$, and so, $\mathbf{v} + \mathbf{e}_i \in X$. Hence, $\mathbf{v} \notin Y$. If $v_i = 1$, then $\mathbf{v} \notin Y$ by definition. In both cases, $\mathbf{v} \notin Y$. Since X is d -closed, $\mathbf{v} \in X$ and therefore, $\mathbf{v} \in X \setminus Y$. Thus, $X \setminus Y$ is d -closed.

Next, we show that $X \setminus Y$ is $(i-1)$ -balanced. Let $\mathbf{a}\mathbf{b} \in X \setminus Y$ with $|\mathbf{a}| = i-1$ and let $\mathbf{c} \in \{0, 1\}^{i-1}$. Since X is $(i-1)$ -balanced, we have that $\mathbf{c}\mathbf{b} \in X$. When $b_1 = 0$, we have by definition of Y that $\mathbf{a}\mathbf{b} + \mathbf{e}_i \in X$ since $\mathbf{a}\mathbf{b} \notin Y$. Since X is $(i-1)$ -balanced and $\mathbf{a}\mathbf{b} + \mathbf{e}_i \in X$, we have that $\mathbf{c}\mathbf{b} + \mathbf{e}_i \in X$ and so, $\mathbf{c}\mathbf{b} \notin Y$. When $b_1 = 1$, we have that $\mathbf{c}\mathbf{b} \notin Y$ by definition. Therefore, since $\mathbf{c}\mathbf{b} \in X$, it follows that $\mathbf{c}\mathbf{b} \in X \setminus Y$. Thus, $X \setminus Y$ is $(i-1)$ -balanced. \square

Finally, we have the following lemma that reduces the induction step to demonstrating inequality (15).

Lemma 26. *If X is an d -closed, $(i - 1)$ -balanced and bad set, and*

$$\Xi_X(Z) \leq \Xi_{X \setminus Y}(Y), \quad (15)$$

then $X \cup Z$ is a d -closed, i -balanced and bad set. In other words, $X \cup Z \in \mathcal{X}_i$.

Proof. By Lemma 24, it remains to show that $X \cup Z$ is a bad set.

Since $X \setminus Y$ is both d -closed and $(i - 1)$ -balanced by Lemma 25, we apply the removal lemma to have that $|Y| > \Xi_{X \setminus Y}(Y)$. Furthermore, we have that $|X| + |Z| = |X \cup Z|$ and $|Z| = |Y|$. Combining these inequalities and equalities, we have that

$$\begin{aligned} |X \cup Z| &= |X| + |Z| \\ &= |X| + |Y| \\ &> |N(X)| + \Xi_{X \setminus Y}(Y) \\ &\geq |N(X)| + \Xi_X(Z) = |N(X \cup Z)|. \end{aligned}$$

Thus, $X \cup Z$ is a bad set. □

In view of Lemma 26, our goal is to show that (15) holds. To this end, let us first introduce more notation.

Definition 8. *For any vector $v \in S$, we let $\Psi(v)$ denote the **additional neighbourhood** of v , i.e.*

$$\Psi(v) \stackrel{\text{def}}{=} |\{u \in N(v) : u \notin N(v') \text{ for all proper descendant } v' \prec v\}|.$$

For a set $V \subseteq S$, define $\Psi(V) = \sum_{v \in V} \Psi(v)$.

The definition of Ψ is useful for computing the size of certain neighbourhoods.

Lemma 27. *For any d -closed set, we have that $|N(V)| = \Psi(V)$.*

Proof. Suppose $V = \{v_1, v_2, \dots, v_m\}$, where $\text{wt}(v_j) \leq \text{wt}(v_{j+1})$ for $j = 1, 2, \dots, m - 1$. That is, we have ordered the vectors in V by weight. Let $V_j \stackrel{\text{def}}{=} \{v_1, v_2, \dots, v_j\}$ for $j = 1, 2, \dots, m$ with the convention that $V_0 = \emptyset$. Then

$$|N(V)| = \sum_{j=1}^m \Xi_{V_{j-1}}(V_j) = \sum_{j=1}^m \Psi(v_j) = \Psi(V).$$

$\Xi_{V_{j-1}}(V_j) = \Psi(v_j)$ follows from the fact that any proper descendant of v_j is contained in V_{j-1} . □

Lemma 28. *Suppose a subset $U \cup V$ of \mathcal{S} is d -closed. Then*

$$\Xi_U(V) = \Psi(V),$$

provided that the sets U and V are disjoint, and U does not contain an ancestor of any vector in V . Hence, if $U \cup V$ and U are d -closed and U and V are disjoint, then $\Xi_U(V) = \Psi(V)$.

Proof. Similar to Lemma 27, we adjoin the vectors in V to U in the order of their weight and define $V_0 = U$ by convention. Then

$$\Xi_U(V) = \sum_{j=1}^m \Xi_{V_{j-1}}(V_j) = \sum_{j=1}^m \Psi(v_j) = \Psi(V).$$

as before. To see that $\Xi_{V_{j-1}}(V_j) = \Psi(v_j)$, we note the following. First, when v_j is adjoined to V_{j-1} , all of its descendants of v_j are already in V_{j-1} since $U \cup V$ is d -closed. Moreover, v_j is not in V_{j-1} and none of its ancestors are in V_{j-1} . □

Lemma 28 then allows us to compute the $\Xi_{X \setminus Y}(Y)$ and $\Xi_X(Z)$ by simply evaluating Ψ .

Corollary 29.

$$\Xi_{X \setminus Y}(Y) = \Psi(Y), \text{ and } \Xi_X(Z) = \Psi(Z).$$

Combining Lemma 26 and Corollary 29, to complete the induction step, it is sufficient to show that

$$\Psi(Z) \leq \Psi(Y). \quad (16)$$

6.1. Equitable domination graphs

We complete the proof of Theorem 1. To this end, we consider an equitable domination graph. If we set $\delta = \lfloor n/m \rfloor$, then the left vertices in the domination graph have degrees δ and $\delta + 1$.

We have the following characterization of $\Psi(v)$ in terms of its weight.

Lemma 30. *For $v \in \{0, 1\}^m$, let the weight of v be ℓ . Then for sufficiently large m (which implies sufficiently large δ), there exist constants c_1, c_2, \dots, c_w such that*

$$\Psi(v) = \begin{cases} c_\ell \delta^w + O(\delta^{w-1}), & \text{if } \ell \leq w, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, when $\text{wt}(v) = w$, we have that $\Psi(v) = \delta^j (\delta + 1)^{w-j} \geq \delta^w$ for some $0 \leq j \leq w$.

Proof. It suffices to consider the case where $\ell \leq w$. Let j be the number of vertices in $\text{supp}(v)$ with degree δ . Then the number of vertices in $\text{supp}(v)$ with degree $\delta + 1$ is $\ell - j$ and we have that

$$\begin{aligned} \Psi(v) &= \sum_{\substack{i_1+i_2+\dots+i_\ell \leq w \\ i_1, i_2, \dots, i_\ell > 0}} \binom{\delta}{i_1} \binom{\delta}{i_2} \cdots \binom{\delta}{i_j} \binom{\delta+1}{i_{j+1}} \binom{\delta+1}{i_{j+2}} \cdots \binom{\delta+1}{i_\ell} \\ &= \sum_{\substack{i_1+i_2+\dots+i_\ell = w \\ i_1, i_2, \dots, i_\ell > 0}} \left(\lambda_{i_1, i_2, \dots, i_\ell} \delta^w + O(\delta^{w-1}) \right) + O(\delta^{w-1}) = c_\ell \delta^w + O(\delta^{w-1}), \end{aligned}$$

where $c_\ell = \sum_{\substack{i_1+i_2+\dots+i_\ell = w \\ i_1, i_2, \dots, i_\ell > 0}} \lambda_{i_1, i_2, \dots, i_\ell}$. When $\ell = w$, the only index for the summand is $i_1 = i_2 = \dots = i_\ell = 1$

and hence, $\Psi(v) = \delta^j (\delta + 1)^{w-j} \geq \delta^w$. Thus, the proof is completed. \square

In addition, we introduce the notion of maximal-support words.

Definition 9. *Given $V \subseteq \mathcal{S}$, we say that $v \in V$ is **maximal-support in V** (or a **maximal-support word of V**) if V does not contain a proper ancestor of v .*

Recall that X is the smallest set in \mathcal{X}_{i-1} . We have the following technical lemmas.

Lemma 31. *Any maximal-support word in X is of the form $\mathbf{1b}$, where $\mathbf{1}$ is the all-ones vector of length $i - 1$.*

Lemma 32. *A maximal-support word in X has weight at least $w + 1$.*

Proof. It suffices to consider the case $i - 1 \leq w$. We prove by contradiction and assume that $\mathbf{u} = \mathbf{1b}$ is a maximal-support word of weight at most w , and where $\mathbf{1}$ is the all-ones vector of length $i - 1$.

Consider the set $U = \{\mathbf{ab} : \mathbf{a} \in \{0, 1\}^{i-1}\}$. We claim that $X \setminus U$ is both $(i - 1)$ -balanced and d-closed. Since both U and X are $(i - 1)$ -balanced, it follows that $X \setminus U$ is $(i - 1)$ -balanced.

On the other hand, consider any word \mathbf{ab} in U and suppose $\mathbf{a'b'}$ $\in X$ is an ancestor of \mathbf{ab} . In other words, $\mathbf{b} \prec \mathbf{b'}$. Since $\mathbf{1b}$ is a maximal-support word, we have that $\mathbf{b'} = \mathbf{b}$ and so, $\mathbf{a'b'} \in U$. Hence, $X \setminus U$ is d-closed.

Since all words in U have weight at most w , $\Psi(\mathbf{v}) \geq 1$ for all $\mathbf{v} \in U$. Therefore, $\Xi_{X \setminus U}(U) = \Psi(U) \geq |U|$, contradicting the removal lemma. \square

Lemma 33. *Given w and $m \geq 2w$, if*

$$\delta^w \geq 2^{2w-1} \sum_{j=0}^{w-1} \binom{m-w}{j} \quad (17)$$

then any word \mathbf{u} of weight at most w is contained in some maximal-support word of weight at least $2w$.

Proof. We prove by contradiction, i.e. suppose that all maximal-support words have weight at most $2w - 1$. From Lemma 32, we have that $\mathbf{u} \prec \mathbf{v}$ for some word \mathbf{v} with weight w . Write $\mathbf{v} = \mathbf{v}_a \mathbf{v}_b$ with $|\mathbf{v}_a| = i - 1$.

As before, we set $U = \{\mathbf{ab} \in X : \mathbf{a} \in \{0, 1\}^{i-1}, \mathbf{v}_b \prec \mathbf{b}\}$. We claim that $X \setminus U$ is both $(i - 1)$ -balanced and d-closed.

Since both U and X are $(i - 1)$ -balanced, it follows that $X \setminus U$ is $(i - 1)$ -balanced.

On the other hand, consider any word \mathbf{ab} in U and suppose $\mathbf{a'b'}$ $\in X$ is an ancestor of \mathbf{ab} . In other words, $\mathbf{b} \prec \mathbf{b'}$ and so, $\mathbf{v}_b \prec \mathbf{b'}$. Therefore, $\mathbf{a'b'} \in U$, and hence, $X \setminus U$ is d-closed.

Next, we provide an upper bound for the set $|U|$. Consider $\mathbf{ab} \in U$. Since all maximal-support words of \mathbf{v} has weight at most $2w - 1$, the weight of \mathbf{b} is at most $2w - i$. Let the weight of \mathbf{v}_b be w_b and so, $w_b \geq w - i + 1$. Since $\mathbf{v}_b \prec \mathbf{b}$, the number of choices for \mathbf{b} is

$$\sum_{j=0}^{2w-i-w_b} \binom{m-w_b-i+1}{j} \leq \sum_{j=0}^{w-1} \binom{m-w}{j}.$$

Hence, $|U| \leq 2^{i-1} \sum_{j=0}^{w-1} \binom{m-w}{j} \leq 2^{2w-1} \sum_{j=0}^{w-1} \binom{m-w}{j}$.

On the other hand, since $X \setminus U$ is d-closed, we have $\Xi_{X \setminus U}(U) = \Psi(U) \geq \Psi(\mathbf{v}) \geq \delta^w$. Then (17) contradicts the removal lemma that states $|U| > \Xi_{X \setminus U}(U)$. \square

Given the existence of such maximal-support words, we now make estimates on $\Psi(Y)$ and $\Psi(Z)$. For convenience, we partition $Y = \bigcup_{\ell=0}^{m-1} Y_\ell$ and $Z = \bigcup_{\ell=1}^m Z_\ell$ such that Y_ℓ and Z_ℓ are words of weight ℓ in Y and Z , respectively.

The following lemma is an immediate consequence from the definition of Y and Z .

Lemma 34. *For $0 \leq \ell \leq m - 1$, $|Y_\ell| = |Z_{\ell+1}|$. The number of words with weight ℓ in Y is equal to the number of words with weight $\ell + 1$ in Z .*

The following lemma is a consequence from Lemma 33.

Lemma 35. *If (17) holds, then for $1 \leq \ell \leq w$, we have that $|Y_\ell| > |Y_{\ell-1}|$.*

Proof. We form a bipartite graph $\mathcal{G} = (Y_\ell \cup Y_{\ell-1}, E)$. Two vertices $v_1 \in Y_\ell$ and $v_2 \in Y_{\ell-1}$ are connected by an edge if $v_2 \prec v_1$. Observe that the degree of a vertex in Y_ℓ is at most ℓ . On the other hand, the degree of a vertex u in $Y_{\ell-1}$ is at least $2w - \ell + 1 > \ell$ since Lemma 33 provides a maximal support word v of weight at least $2w$ such that $u \prec v$. Since the sum of degrees in Y_ℓ is equal to the sum of degrees in $Y_{\ell-1}$ and each vertex in Y_ℓ has a smaller degree than a vertex in $Y_{\ell-1}$, it follows that $|Y_\ell| > |Y_{\ell-1}|$. \square

Corollary 36. *If (17) holds, then for $1 \leq \ell \leq w$, we have that $|Y_\ell| > |Z_\ell|$.*

We now prove the main result on asymptotic existence.

Proof of Theorem 1. Applying Lemma 30, we have that

$$\Psi(Y) = \sum_{\ell=0}^m \sum_{v \in Y_\ell} \Psi(v) = \sum_{\ell=0}^w |Y_\ell| \left(c_\ell \delta^{2w} + O(\delta^{2w-1}) \right) \geq \sum_{\ell=1}^w |Y_\ell| \left(c_\ell \delta^{2w} + O(\delta^{2w-1}) \right).$$

Similar manipulations yield

$$\Psi(Z) = \sum_{\ell=1}^w |Z_\ell| \left(c_\ell \delta^{2w} + O(\delta^{2w-1}) \right).$$

Since $|Y_\ell| - |Z_\ell| \geq 1$, we estimate the difference $\Psi(Y) - \Psi(Z)$ by

$$\Psi(Y) - \Psi(Z) \geq \sum_{\ell=1}^w c_\ell \delta^{2w} + O(\delta^{2w-1}). \quad (18)$$

Therefore, for fixed values of w , if we choose m sufficiently large such that (17) holds and the right hand side(18) is nonnegative, we have that $\Psi(Z) \leq \Psi(Y)$. In other words, we establish (16) and complete our induction argument. Hence, no bad sets exist in the associated compatibility graph and a perfect matching or an (m, n, w) -domination mapping exists. \square

Unfortunately, estimating the values of c_ℓ is difficult and hence, we are unable to estimate a lower bound of m for which domination mapping exists. Nevertheless, for the remaining part of the section, we consider the case when m divides n and demonstrate that the requirement defined by (17) is mild. In this case, the domination graph is regular where each vertex in $[m]$ has degree exactly δ . Then it follows from symmetry that $\Psi(v)$ is dependent only on the weight of v . In other words, for any word v of weight ℓ , we can write $\Psi(v)$ as Ψ_ℓ . Therefore,

$$\Psi(Y) = \sum_{\ell=0}^w |Y_\ell| \Psi_\ell \geq \sum_{\ell=1}^w |Y_\ell| \Psi_\ell > \sum_{\ell=1}^w |Z_\ell| \Psi_\ell = \Psi(Z).$$

Hence, we have the following corollary.

Corollary 37. *If (17) holds, then $\Psi(Z) < \Psi(Y)$.*

The next theorem provide certain sufficient numerical conditions for the existence of $(m, \delta m, w)$ -domination mappings and imply Theorem 1.

Theorem 38. *Let $w \geq 3$. Let $\epsilon > 0$ and set N_ϵ such that $m^{\epsilon/(1+\epsilon)} \geq 1 + \log m$ for $m \geq N_\epsilon$. If*

$$m \geq \max \left\{ (2w)^{1+\epsilon}, N_\epsilon \right\}, \quad (19)$$

$$2^m \leq \sum_{j=0}^w \binom{\delta m}{j}, \quad (20)$$

then (17) holds and therefore, an $(m, \delta m, w)$ -domination mapping exists.

Proof. Since $m \geq 2w$, observe that $\sum_{j=0}^w \binom{\delta m}{j} \leq (w+1)\binom{\delta m}{w} = (w+1)(\delta m)^w/w!$. Hence, (20) implies that

$$\delta^w \geq 2^m \frac{(w+1)!}{wm^w}. \quad (21)$$

On the other hand, (19) implies that

$$m = m^{\frac{1}{1+\epsilon}} m^{\frac{\epsilon}{1+\epsilon}} \geq 2w(1 + \log m) \geq 2w \log(2m).$$

Hence, $2^m \geq (2m)^{2w}$. Together with (21), we have

$$\delta^w \geq 2^{2w} \frac{w!m^w}{(w+1)} \geq 2^{2w-1} \frac{wm^w}{(w-1)!} \geq 2^{2w-1} w \binom{m-w}{w-1} \geq 2^{2w-1} \sum_{j=0}^{w-1} \binom{m-w}{j}.$$

Note that $2w!/(w+1) \geq w/(w-1)!$ for $w \geq 3$. Therefore, (17) holds as desired. Corollary 37 then yields (16), which in turn completes our induction argument. \square

A. Constructions and Descendant Arrays

A.1. Descendant arrays

We will now make use of a different representation of (m, n, w) -domination mappings. For two vectors of the same length v and u we say that v *covers* u (or v *dominates* u ; or u is a *descendant* of v) and denote it by $u \prec v$ if u has a *zero* in each entry in which v has a *zero*. In other words, the value of u in each position is less or equal from the value of v in the same position. Let A be an $r \times \ell_1$ binary array and B be an $r \times \ell_2$ binary array, $\ell_2 \geq \ell_1$. If for each column in B there exists a column in A which dominates it, then (A, B) is called a pair of (ℓ_1, ℓ_2) -*descendant arrays*. Let A be a $2^m \times m$ matrix which contains all the binary words of length m (in lexicographic order) and let B be a $2^m \times n$ matrix which contains distinct binary words of length n and weight at most w . This pair of arrays is called a pair of (n, m, w) -*descendant arrays* if (A, B) is a pair of (n, m) -*descendant arrays*. By definition we infer that

Lemma 39. *An (m, n, w) -domination mapping exists if and only if there exists a pair of (m, n, w) -descendant arrays.*

Proof. This is an immediate observation from the definition of the domination graph and defining $\varphi(v_1, v_2, \dots, v_m) = (u_1, u_2, \dots, u_n)$, where (v_1, v_2, \dots, v_m) and (u_1, u_2, \dots, u_n) are the j th words of the matrices A and B , respectively. \square

The description which leads to Lemma 39 can be used as an alternative way to define the injective mapping $\varphi(m, n, w)$ and to verify whether such a mapping is a domination mapping.

A.2. Construction of domination mappings for $w = 2$

In this appendix we consider $(m, n, 2)$ -domination mappings for which $n = \nu(m, 2)$ if m is odd and discuss the case of even m . We start with odd m . We claim that for odd $m = 2\ell + 1$ there exists a $(2\ell + 1, 2^{\ell+1}, 2)$ -domination mapping. Note, that $\sum_{i=0}^2 \binom{2^{\ell+1}}{i} = 2^{2\ell+1} + 2^\ell + 1$ and $\sum_{i=0}^2 \binom{2^{\ell+1}-1}{i} = 2^{2\ell+1} - 2^\ell + 1$, and hence by Lemma 5 we have that a $(2\ell + 1, 2^{\ell+1}, 2)$ -domination mapping is optimal. A recursive construction for such a domination mapping will be given. We start with the $(3, 4, 2)$ -domination mapping from Example 1.

Assume now that there exists a $(2\ell - 1, 2^\ell, 2)$ -domination mapping φ with the degree sequence $(1, 1, 2, 2, 2, 4, 4, 8, 8, \dots, 2^{\ell-3}, 2^{\ell-3}, 2^{\ell-2}, 2^{\ell-2})$. We will construct a $(2\ell + 1, 2^{\ell+1}, 2)$ -domination mapping φ' with the degree sequence $(1, 1, 2, 2, 2, 4, 4, 8, 8, \dots, 2^{\ell-2}, 2^{\ell-2}, 2^{\ell-1}, 2^{\ell-1})$.

Let $(A(\varphi), B(\varphi))$ be a pair of $(2\ell - 1, 2^\ell, 2)$ -descendant arrays related to the domination mapping φ . We will describe now a construction for a pair of $(2\ell + 1, 2^{\ell+1}, 2)$ -descendant arrays $(A(\varphi'), B(\varphi'))$ from which φ' can be derived. Let A_1A_2 be a $2^{2\ell-1} \times (2\ell + 1)$ matrix, where A_1 has two columns and A_2 has $2\ell - 1$ columns; A_1A_2 represents a quarter of the matrix $A(\varphi')$, i.e. A_1 has one of the four values 00, 01, 10, or 11, and A_2 has $2^{2\ell-1}$ rows with exactly all the words in $\mathbb{F}_2^{2\ell-1}$ in the lexicographic order. Let B_1B_2 be a $2^{2\ell-1} \times 2^{\ell+1}$ matrix, where B_1 has 2^ℓ columns and B_2 has 2^ℓ columns; B_1B_2 represents a quarter of the matrix $B(\varphi')$. Furthermore, let $B_1 = B_1^0B_1^1$, where B_1^0 and B_1^1 are $2^{2\ell-1} \times 2^{\ell-1}$ matrices. The i th row of A_1A_2 will be mapped by φ' to the i th row of B_1B_2 . We distinguish now between four cases related to the values of the two columns of A_1 .

[D1] If the two columns of A_1 are 00, then B_1 is the all-zeroes matrix and (A_2, B_2) is a pair of $(2\ell - 1, 2^\ell, 2)$ -descendant arrays with the degree sequence $(1, 1, 2, 2, 2, 4, 4, 8, 8, \dots, 2^{\ell-3}, 2^{\ell-3}, 2^{\ell-2}, 2^{\ell-2})$.

[D2] If the two columns of A_1 are 11, then B_2 is the all-zeroes matrix, and B_1 can be any matrix whose rows are different, $2^{2\ell-1} - 2^{\ell-1}$ rows have weight two, $2^{\ell-1}$ rows have weight one ($2^{\ell-2}$ of these *ones* in the last $2^{\ell-2}$ columns of B_1^0 and the other $2^{\ell-2}$ *ones* in the last $2^{\ell-2}$ columns, which are the most significant bits, of B_1^1). Simple enumeration yields that there are $2^{2\ell-1}$ such possible different rows as required.

[D3] If the two columns of A_1 are 01 then the matrix B_2 is chosen in a way that each column, except for the first one (least significant bit) has exactly $2^{\ell-1}$ *ones* and the first column has $2^{\ell-2}$ *ones*; each row, except for $2^{\ell-2}$ rows of B_2 , has exactly one *one* and these $2^{\ell-2}$ rows are all-zeroes rows. The distribution of the *ones* is done in a way that the requirements of the related domination graph, i.e. the degree sequence of (A_2, B_2) as a pair of $(2\ell - 1, 2^\ell)$ -descendant arrays is $(1, 1, 2, 2, 2, 4, 4, 8, 8, \dots, 2^{\ell-3}, 2^{\ell-3}, 2^{\ell-2}, 2^{\ell-2})$. For each $2^{\ell-1}$ *ones* in the same column of B_2 , the corresponding $2^{\ell-1}$ rows in B_1 have unique *ones* in different $2^{\ell-1}$ columns in the last $2^{\ell-1}$ columns of B_1 . The same is done for the first column of B_2 for which the all-zeroes rows are added. The concrete definition will be left as an exercise (not completely trivial).

[D4] If the two columns of A_1 are 10 then B_2 is exactly as in the case where the two columns of A_1 are 01, In B_1 the first $2^{\ell-1}$ columns are swapped with the last $2^{\ell-1}$ columns compared to the matrix B_1 in the case where the first two columns of A_1 are 01, i.e. B_1^0 is swapped with B_1^1 .

Finally, $A(\varphi')$ consists of the four matrices A_1A_2 of these four cases and $B(\varphi')$ are the related four matrices B_1B_2 . The construction leads to the following result.

Theorem 40. *If $(A(\varphi), B(\varphi))$ is a pair of $(2\ell - 1, 2^\ell, 2)$ -descendant arrays then $(A(\varphi'), B(\varphi'))$ are two $(2\ell + 1, 2^{\ell+1}, 2)$ -descendant arrays.*

Theorem 40 implies the existence of a $(2\ell + 1, 2^{\ell+1}, 2)$ -domination mapping for odd m . What about an $(m, n, 2)$ -domination mapping for even m . The following (m, n) pairs were found by computer search to form optimal $(m, n, 2)$ -domination mappings: $(4, 6)$, $(6, 11)$, $(8, 23)$, $(10, 45)$, $(12, 90)$, $(14, 181)$, $(16, 362)$, $(18, 724)$, $(20, 1448)$, $(22, 2896)$, $(24, 5793)$, $(26, 11585)$, $(28, 23170)$, and so on. For $m \geq 6$, these mappings attains the bound of Lemma 5. In this sequence of optimal domination mappings one can observe that

there is no obvious rule and a hence a recursive construction for optimal domination mapping won't be an easy task. But, optimal domination mappings exist and an existence proof for such mappings can be given similarly (but with a simpler proof) to the existence proof in Section 6.

B. Proof of Proposition 15

We provide a detailed proof of Proposition 15. To this end, we need the following lemmas.

Lemma 41. *The set of all A -preserving permutations is a subgroup of the set of permutations on $[N]$.*

Lemma 42. *Suppose that π is A -preserving. Then $Ax^\pi \leq \mathbf{1}$ if and only if $Ax \leq \mathbf{1}$.*

Proof. Since the set of all A -preserving permutations form a group, it suffices to show one direction.

Suppose that $Ax \leq \mathbf{1}$. Since $P_{\pi_{\text{row}}}AP_\pi = A$ for some permutation $\pi_{\text{row}} : [M] \rightarrow [M]$, we have that

$$\mathbf{1} \geq Ax = P_{\pi_{\text{row}}}AP_\pi x = P_{\pi_{\text{row}}}Ax^\pi.$$

Hence, $Ax^\pi \leq P_{\pi_{\text{row}}}^{-1}\mathbf{1} = \mathbf{1}$. □

Proof of Proposition 15. Set

$$\mathbf{x}^* = \frac{\sum_{\pi \in G_A} \mathbf{x}^\pi}{|G_A|}.$$

By Lemma 42, we have that $Ax^\pi \leq \mathbf{1}$ for all $\pi \in G_A$. Therefore, $A\mathbf{x}^* \leq \mathbf{1}$. It is also readily verified that $\sum_{i=1}^N x_i^* = \lambda$

Finally, we show that \mathbf{x}^* is \mathcal{O} -regular. For all $k \leq N$ and $i, j \in O_k$, we have that

$$x_i^* = \frac{\sum_{\pi \in G_A} x_i^\pi}{|G_A|} = \frac{\sum_{\pi \in G_A} x_{\pi(i)}}{|G_A|}.$$

On the other hand, since $i, j \in O_k$, there exists a π^* such that $\pi^*(j) = i$. Hence,

$$x_j^* = \frac{\sum_{\pi \in G_A} x_j^\pi}{|G_A|} = \frac{\sum_{\pi \in G_A} x_{\pi(j)}}{|G_A|} = \frac{\sum_{\pi \in G_A} x_{\pi \circ \pi^*(j)}}{|G_A|} = \frac{\sum_{\pi \in G_A} x_{\pi(i)}}{|G_A|}.$$

Therefore, $x_i^* = x_j^*$. □

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