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Arboricity: An acyclic hypergraph decomposition problem motivated by database theory

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ABSTRACT

The arboricity of a hypergraph $\mathscr{H}$ is the minimum number of acyclic hypergraphs that partition $\mathscr{H}$. The determination of the arboricity of hypergraphs is a problem motivated by database theory. The exact arboricity of the complete k-uniform hypergraph of order n is previously known only for $k \in \{1, 2, n-2, n-1, n\}$. The arboricity of the complete k-uniform hypergraph of order n is determined asymptotically when $k = n - O(\log^{1-\delta} n)$, δ positive, and determined exactly when $k = n - 3$. This proves a conjecture of Wang (2008) [20] in the asymptotic sense.

1. Introduction

Acyclic hypergraphs were introduced as a hypergraph analogue of trees in graphs (see Berge [3]). Besides being mathematically interesting, acyclic hypergraphs feature prominently in the study of problems in database theory and constraint satisfaction. There is a natural bijection between database schemas and hypergraphs, where each attribute of a database schema D corresponds to a vertex in a hypergraph $\mathscr{H}$ and each relation R of attributes in D corresponds to an edge in $\mathscr{H}$.

Acyclic database schemas (which correspond to acyclic hypergraphs) were first studied by Beeri et al. [1]. This natural class of database schemas has been shown to possess important and desirable properties [1,9,8,2,7]. Acyclic hypergraphs have since become objects of study by many researchers. One of the primary reasons for the desirability of acyclicity in database schemas is that there are important problems that are NP-hard on general database schemas but which becomes polynomial-time solvable when restricted to acyclic instances. Examples of such problems include the following.

(i) Determining global consistency [2].
(ii) Evaluating conjunctive queries [22].
(iii) Computing joins or projections of joins [22].

Furthermore, acyclic database schemas can be recognized in linear time [18].

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It is therefore natural to decompose an instance into acyclic instances. This has motivated some recent study into the arboricity of a hypergraph (the minimum number of acyclic hypergraphs into which the edges of the given hypergraph can be partitioned).

The main contributions of this paper are the asymptotic determination of the arboricity of complete uniform hypergraphs with large edge size, and the exact determination of the arboricity of an infinite family of complete uniform hypergraphs.

2. Mathematical preliminaries

Let n be a positive integer. The set $\{1, \ldots, n\}$ is denoted $[n]$.

For X a finite set and k a nonnegative integer, the set of all k-subsets of X is denoted by $\binom{X}{k}$, that is,

$$\binom{X}{k} = \{K \subseteq X : |K| = k\}.$$

A *hypergraph* is a pair $\mathcal{H} = (X, \mathcal{A})$, where X is a finite set, and $\mathcal{A} \subseteq 2^X$. The elements of X are called *vertices* and the elements of $\mathcal{A}$ are called *edges*. The *order* of $\mathcal{H}$ is the number of vertices in X, and the *size* of $\mathcal{H}$ is the number of edges in $\mathcal{A}$. If $\mathcal{A} \subseteq \binom{X}{k}$, then $(X, \mathcal{A})$ is said to be *k-uniform*. Note that our usual notion of a graph is equivalent to the notion of a 2-uniform hypergraph. The *complete k-uniform hypergraph* $\left(X, \binom{X}{k}\right)$ *of order n* is denoted $K_n^{(k)}$. A hypergraph is *empty* if it has no edges. The *degree* of a vertex v is the number of edges containing v.

There are many definitions for the *acyclicity* of a hypergraph (see [2]). The definition we use here is based on the *Graham reduction* [10], described below.

Let $\mathcal{H} = (X, \mathcal{A})$ be a given hypergraph. *Graham's algorithm* applies the following operations repeatedly to $\mathcal{H}$ until neither can be applied:

(a) If a vertex $x \in X$ has degree one, then delete x from the edge containing it.
(b) If $A, B \in \mathcal{A}$ are distinct edges such that $A \subseteq B$, then delete A from $\mathcal{A}$.
(c) If $A \in \mathcal{A}$ is empty, then delete A from $\mathcal{A}$.

The resulting hypergraph $\mathcal{H}'$ is said to be *Graham-reduced*, and is called the *Graham reduction* of $\mathcal{H}$.

Definition 2.1. A hypergraph is *acyclic* if its Graham reduction is empty.

Example 2.1. Let $\mathcal{H} = (X, \mathcal{A})$ be the hypergraph such that

$$X = \{1, 2, 3, 4, 5, 6\},$$
$$\mathcal{A} = \{\{1, 2, 3\}, \{3, 4, 5\}, \{1, 5, 6\}, \{1, 3, 5\}\}.$$

We apply operations (a) and (b) repeatedly. The vertices 2, 4, 6 each has degree one and hence can be deleted. We are therefore left with edges $\{1, 3\}$,$\{3, 5\}$,$\{1, 5\}$,$\{1, 3, 5\}$. The edges $\{1, 3\}$,$\{3, 5\}$,$\{1, 5\}$ can be deleted since each is a subset of the edge $\{1, 3, 5\}$. We are then left with the edge $\{1, 3, 5\}$. Now observe that vertices 1, 3, 5 can be deleted since each of them has degree one. The Graham reduction of $\mathcal{H}$ is therefore empty. Consequently, $\mathcal{H}$ is acyclic.

Example 2.2. Let $\mathcal{H} = (X, \mathcal{A})$ be the hypergraph such that

$$X = \{1, 2, 3, 4\},$$
$$\mathcal{A} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}.$$

The Graham reduction of $\mathcal{H}$ is the hypergraph itself since neither operations (a) nor (b) can be applied. $\mathcal{H}$ is therefore *not* acyclic.

Note that acyclicity of hypergraphs coincides with our usual notion of acyclicity in graphs when the hypergraphs under consideration are ordinary graphs.

The following result on the maximum size of a k-uniform acyclic hypergraph is known.

Proposition 2.1 (*Wang and Li [21]*)**.** *A maximum k-uniform acyclic hypergraph of order n has size* $n - k + 1$.

An *acyclic decomposition* of a hypergraph $\mathcal{H} = (X, \mathcal{A})$ is a set of acyclic hypergraphs $\{(X, \mathcal{A}_i)\}_{i=1}^{c}$ such that the following conditions hold:

(i) $\mathcal{A}_i \subseteq \mathcal{A}$ for all $i \in [c]$.
(ii) $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for all distinct $i, j \in [c]$.
(iii) $\cup_{i=1}^{c} \mathcal{A}_i = \mathcal{A}$.

The *size* of the acyclic decomposition is c, the number of acyclic hypergraphs in the decomposition.

Definition 2.2. The *arboricity* of a hypergraph $\mathcal{H}$, denoted $\mathrm{arb}(\mathcal{H})$, is the minimum size of an acyclic decomposition of $\mathcal{H}$.

3. Previous and related work

Trivially, $\mathrm{arb}(K_n^{(1)}) = \mathrm{arb}(K_n^{(n)}) = 1$, since both $K_n^{(1)}$ and $K_n^{(n)}$ are acyclic.

Determining $\mathrm{arb}(K_n^{(2)})$ is closely related to the classical problem of determining $\sigma(K_n)$, the spanning tree packing number of the complete graph K_n (see [16]). First note that $\mathrm{arb}(K_n^{(2)}) \geq \lceil n/2 \rceil$ since a maximum acyclic graph is a spanning tree and has size $n-1$. When n is even, K_n has a decomposition into $n/2$ Hamiltonian paths, and hence $\mathrm{arb}(K_n^{(2)}) = \sigma(K_n) = n/2$ in this case. When n is odd, consider a decomposition of K_{n-1} into $(n-1)/2$ Hamiltonian paths. Now add a new vertex v and from each Hamiltonian path of the decomposition add an edge from a distinct vertex to v. The remaining edges from vertices not already connected to v form a star. This gives a decomposition of K_n into $(n+1)/2$ acyclic graphs ($(n-1)/2$ of which are spanning trees and one of which is a star of size $(n-1)/2$). It follows that $\mathrm{arb}(K_n^{(2)}) \leq \lceil n/2 \rceil$, and consequently $\mathrm{arb}(K_n^{(2)}) = \lceil n/2 \rceil$.

It is also easy to see that $\mathrm{arb}(K_n^{(n-1)}) = \lceil n/2 \rceil$ since any set of at most two edges in $K_n^{(n-1)}$ is acyclic.

Li [13] proved that $\mathrm{arb}(K_n^{(n-2)}) = \lceil n(n-1)/6 \rceil$.

The value of $\mathrm{arb}(K_n^{(k)})$ is unknown for all other k.

The known results on $\mathrm{arb}(K_n^{(k)})$ led Wang [20, ch. 10] to make the following conjecture.

Conjecture 3.1.

$$\mathrm{arb}(K_n^{(k)}) = \left\lceil \frac{1}{k} \binom{n}{k-1} \right\rceil.$$

Li [13] proved that

$$\left\lceil \frac{1}{k} \binom{n}{k-1} \right\rceil \leq \mathrm{arb}(K_n^{(k)}) \leq \frac{1}{2} \binom{n+1}{k-1}. \tag{1}$$

The upper and lower bounds in (1) are approximately a factor of $k/2$ apart.

By substituting $n-k$ for k and simplifying, Conjecture 3.1 can be stated equivalently as.

Conjecture 3.2.

$$\mathrm{arb}(K_n^{(n-k)}) = \left\lceil \frac{1}{k+1} \binom{n}{k} \right\rceil. \tag{2}$$

In the next section, we prove Conjecture 3.2 in the asymptotic sense, if $k = O(\log^{1-\delta} n)$, δ a positive constant, by showing that under this condition, we have

$$\mathrm{arb}(K_n^{(n-k)}) = (1+o(1)) \frac{1}{k+1} \binom{n}{k}.$$

4. An asymptotic result

A k-uniform hypergraph $(X, \mathcal{A})$ of size c is called a *delta-system* $\Delta(p, k, c)$ if there is a set F, called the *center*, such that $|F| = p$ and $A_i \cap A_j = F$ for all distinct $A_i, A_j \in \mathcal{A}$.

Proposition 4.1. *A delta-system $\Delta(p, k, c)$ is acyclic.*

Proof. Let $(X, \mathcal{A})$ be a delta-system $\Delta(p, k, c)$ with center F. Then for each edge $A \in \mathcal{A}$, each of the vertices in $A \setminus F$ is contained in no other edge, for otherwise there would exist some other edge whose intersection with A is not F. We can therefore apply operation (a) of Graham's algorithm to remove the vertices in $A \setminus F$, for each edge $A \in \mathcal{A}$. What remains is a set of edges, each of which is F. Applying operation (b) of Graham's algorithm now gives an empty hypergraph. □

The following is immediate from Proposition 2.1 and Proposition 4.1.

Corollary 4.1. *A delta-system $\Delta(k-1, k, n-k+1)$ is a maximum k-uniform acyclic hypergraph of order n.*

Definition 4.1. Let $\mathcal{H} = (X, \mathcal{A})$ be a hypergraph. The *supplement* of $\mathcal{H}$, denoted $\mathcal{H}^s$, is the hypergraph $(X, \mathcal{B})$, where $\mathcal{B} = \{X \setminus A : A \in \mathcal{A}\}$.

Note that $(\mathcal{H}^s)^s = \mathcal{H}$.

Theorem 4.1. *Let $\mathcal{H} = (X, \mathcal{A})$ be a k-uniform hypergraph of order n. Then $\mathcal{H}$ is a delta-system $\Delta(k-1, k, n-k+1)$ if and only if the edges of $\mathcal{H}^s$ induce a $K_{n-k+1}^{(n-k)}$.*

Proof. First note that since $\mathcal{H}$ is k-uniform, $\mathcal{H}^s$ is $n-k$-uniform. Next, observe that the only degree zero vertices in $\mathcal{H}^s$ are those that appear in every edge of $\mathcal{H}$. In particular, the set of degree zero vertices in $\mathcal{H}^s$ is precisely F, the center of $\mathcal{H}$. Hence, the edges of $\mathcal{H}^s$ induce a subgraph of order $n-k+1$.

Finally, observe that the edge containing $x \in X \setminus F$ in $\mathcal{H}$ gives rise to the edge $X \setminus (F \cup \{x\})$ in $\mathcal{H}^s$, and vice versa. This completes the proof. □

We now establish an asymptotically exact value for $\mathrm{arb}(K_n^{(n-k)})$, when $k = O(\log^{1-\delta} n)$, δ a positive constant. First, we require some concepts and results from combinatorial design theory.

Definition 4.2. A t-$(v, k, 1)$ *packing* is a k-uniform hypergraph $(X, \mathcal{A})$ of order v such that every t-subset of X is contained in at most one edge of $\mathcal{A}$.

Given t, k, and v, the determination of $D(t, k, v)$, the maximum size of a t-$(v, k, 1)$ packing, constitutes a central problem in combinatorial design theory, as well as in coding theory [15]. It is easy to see that $D(t, k, v) \le \binom{v}{t} / \binom{k}{t}$. Rödl [17] was the first to show that this upper bound can be attained asymptotically. Let $\epsilon_{t,k}(v)$ be the fraction of t-subsets not contained in any edges of a t-$(v, k, 1)$ packing of maximum size. In other words, $D(t, k, v) = (1 - \epsilon_{t,k}(n)) \binom{v}{t} / \binom{k}{t}$.

Theorem 4.2 (Rödl [17]). *For fixed t and k, we have* $\lim_{v\to\infty} \epsilon_{t,k}(v) = 0$.

The best current bound on $\epsilon_{t,k}(v)$ is by Vu [19], who showed that

$$\epsilon_{t,k}(v) = O(v^{-\beta} \log^{\gamma} v), \tag{3}$$

where $\beta = 1/(\binom{k}{t} - 1)$, and $\gamma > 0$ is a constant.

Let $(X, \mathcal{A})$ be a k-$(n, k+1, 1)$ packing of maximum size, so that $|A| = (1 - \epsilon_{k,k+1}(n)) \frac{1}{k+1} \binom{n}{k}$. The number of k-subsets of X not contained in any edges of $\mathcal{A}$ is $\epsilon_{k,k+1}(n) \binom{n}{k}$. Now, for each edge $A \in \mathcal{A}$, let $\mathcal{B}_A$ be the set of all k-subsets of A, that is, $\mathcal{B}_A = \binom{A}{k}$. Further, let $\mathcal{B}'$ be the set of all k-subsets of X not contained in any edges of $\mathcal{A}$. Let $\mathcal{B} = (\cup_{A\in\mathcal{A}} \mathcal{B}_A) \cup (\cup_{B\in\mathcal{B}'} \{B\})$. Then $\mathcal{B} = \binom{X}{k}$, since every k-subset of X is either contained in $\mathcal{B}_A$ for exactly one $A \in \mathcal{A}$, or is contained in $\mathcal{B}'$. Consider the supplement $\mathcal{H}^s = (X, \mathcal{C})$ of the complete k-uniform hypergraph $\mathcal{H} = (X, \mathcal{B})$ of order n. It is clear that $\mathcal{H}^s$ is a complete $(n-k)$-uniform hypergraph of order n. Since $\{\mathcal{B}_A\}_{A\in\mathcal{A}}$ together with $\{\{B\}\}_{B\in\mathcal{B}'}$ partition the edge set of $\mathcal{B}$, then their supplements also partition the edge set of $\mathcal{C}$. We now examine the structure of the supplement of $(X, \mathcal{B}_A)$ for $A \in \mathcal{A}$, and $(X, \{B\})$ for $B \in \mathcal{B}'$. Observe that, ignoring the degree zero vertices, $(X, \mathcal{B}_A)$ is a $K_{k+1}^{(k)}$. Applying Theorem 4.1 shows that its supplement is a delta-system $\Delta(n-k-1, n-k, k+1)$, which is acyclic by Corollary 4.1. The supplement of $(X, \{B\})$ contains just a single edge, and is therefore also acyclic. The supplements of $(X, \mathcal{B}_A)$ and $(X, \{B\})$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}'$, is therefore an acyclic decomposition of $\mathcal{H}^s$. The size of this acyclic decomposition is

$$\begin{aligned} |\mathcal{A}| + |\mathcal{B}'| &= (1 - \epsilon_{k,k+1}(n)) \frac{1}{k+1} \binom{n}{k} + \epsilon_{k,k+1}(n) \binom{n}{k} \\ &= (1 + k\epsilon_{k,k+1}(n)) \frac{1}{k+1} \binom{n}{k}. \end{aligned}$$

From (3), we have

$$k\epsilon_{k,k+1}(n) = O\left(\frac{k \log^{\gamma} n}{n^{1/k}}\right),$$

for some positive constant γ. Thus, $\lim_{n\to\infty} k\epsilon_{k,k+1}(n) = 0$ when $k = O(\log^{1-\delta} n)$, for any constant $\delta > 0$.

We summarize the above discussions as:

Theorem 4.3. *Let δ be a positive constant. Then for $k = O(\log^{1-\delta} n)$, we have*

$$\mathrm{arb}(K_n^{(n-k)}) = (1 + o(1)) \frac{1}{k+1} \binom{n}{k}.$$

If in a t-$(v, k, 1)$ packing $(X, \mathcal{A})$, every t-subset of X is contained in exactly one (instead of at most one) edge of $\mathcal{A}$, the packing is known as a *Steiner system*, and is denoted $S(t, k, v)$. Whenever an $S(k, k+1, n)$ exists, then the discussions above show that $K_n^{(n-k)}$ has an acyclic decomposition into delta-systems $\Delta(n-k-1, n-k, k+1)$, which are maximum acyclic hypergraphs, so that we have.

Theorem 4.4. *Let $k < n$ be positive integers. If there exists a Steiner system $S(k, k+1, n)$, then*

$$\mathrm{arb}(K_n^{(n-k)}) = \frac{1}{k+1}\binom{n}{k}.$$

Corollary 4.2. *When any one of the conditions*

(i) $k = 1$ *and* $n \equiv 0 \pmod 2$,
(ii) $k = 2$ *and* $n \equiv 1$ *or* $3 \pmod 6$,
(iii) $k = 3$ *and* $n \equiv 2$ *or* $4 \pmod 6$,
(iv) $k = 4$ *and* $n \in \{11, 23, 35, 47, 71, 83, 107, 131\}$,
(v) $k = 5$ *and* $n \in \{12, 24, 36, 48, 72, 84, 108, 132\}$,

is satisfied, we have

$$\mathrm{arb}(K_n^{(n-k)}) = \frac{1}{k+1}\binom{n}{k}.$$

Proof. For (i), note that an $S(1, 2, n)$ is a perfect matching in the complete graph K_n, and hence exists if and only if n is even. For (ii), an $S(2, 3, n)$ is a Steiner triple system and exists if and only if $n \equiv 1$ or $3 \pmod 6$ (see, for example, [6]). For (iii), an $S(3, 4, n)$ is a Steiner quadruple system, existence for which was settled by Hanani [12], who showed that $n \equiv 2$ or $4 \pmod 6$ is necessary and sufficient. For (iv)–(v), see [11,5] for existence results. □

5. Arboricity of $K_n^{(n-3)}$

The purpose of this section is to determine the exact value of $\mathrm{arb}(K_n^{(n-3)})$ completely. Corollary 4.2(iii) already gives

$$\mathrm{arb}(K_n^{(n-3)}) = \frac{n(n-1)(n-2)}{24}$$

when $n \equiv 2$ or $4 \pmod 6$, so we focus on the remaining cases of $n \equiv 0, 1, 3$ or $5 \pmod 6$.

We need more combinatorial constructs.

Definition 5.1. Let t, k, m, and v be nonnegative integers. A (t, k) *candelabra system* of order v is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

(i) $(X, \mathcal{A})$ is a k-uniform hypergraph of order v.
(ii) $S \subseteq X$, called the *stem*.
(iii) $\mathcal{G} = \{G_1, \ldots, G_m\}$ is a partition of $X \setminus S$ (elements of $\mathcal{G}$ are called *groups*).
(iv) Every t-subset $T \subseteq X$ with $|T \cap (S \cup G_i)| < t$ for all $i \in [m]$ is contained in exactly one edge of $\mathcal{A}$, and no t-subsets of $S \cup G_i$, for any $i \in [m]$, is contained in an edge of $\mathcal{A}$.

The *type* of a (t, k) candelabra system $(X, S, \mathcal{G}, \mathcal{A})$ is the multiset $[|G| : G \in \mathcal{G}]$. For convenience, the type is often written with the exponential notation so that the type $g_1^{s_1} g_2^{s_2} \ldots g_r^{s_r}$ is taken to mean that there are s_i groups of size g_i, for $i \in [r]$. A (t, k) candelabra system of type $g_1^{s_1} g_2^{s_2} \ldots g_r^{s_r}$ with a stem of size s is denoted by (t, k)-CS$(g_1^{s_1} g_2^{s_2} \ldots g_t^{s_t} : s)$, and has order $\sum_{i=1}^{t} g_i s_i + s$.

Mills [14] constructed an infinite class of (3,4) candelabra systems.

Proposition 5.1 (*Mills [14]*)**.** *For all $m \geq 0$, there exists a (3,4)-CS$(6^m : 0)$ of order $6m$.*

5.1. *The case $n \equiv 0 \pmod 6$*

We begin by showing that $\mathrm{arb}(K_6^{(3)}) = 5$. Consider the complete 3-uniform hypergraph $\left(X, \binom{X}{3}\right)$, where $X = [6]$. Let

$$\mathcal{A}_1 = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\},$$
$$\mathcal{A}_2 = \{\{1,2,5\}, \{1,2,6\}, \{1,5,6\}, \{2,5,6\}\},$$
$$\mathcal{A}_3 = \{\{3,4,5\}, \{3,4,6\}, \{3,5,6\}, \{4,5,6\}\},$$
$$\mathcal{A}_4 = \{\{1,3,5\}, \{2,3,5\}, \{2,3,6\}, \{2,4,6\}\},$$
$$\mathcal{A}_5 = \{\{1,3,6\}, \{1,4,5\}, \{1,4,6\}, \{2,4,5\}\}.$$

It is easy to verify that the supplement of $(X, \mathcal{A}_i)$, $i \in [5]$, is acyclic. Hence, these supplements give an acyclic decomposition of $\left(X, \binom{X}{3}\right)$, proving $\mathrm{arb}(K_6^{(3)}) = 5$.

Now, let $n \equiv 0 \pmod 6$ and let $(X, S, \mathcal{G}, \mathcal{A})$ be a (3,4)-CS$(6^{n/6} : 0)$ of order n, which exists by Proposition 5.1. Any 3-subset of X not contained in a group is contained in exactly one edge of $\mathcal{A}$. The number of such 3-subsets is

$$\binom{n}{3} - \frac{n}{6}\binom{6}{3}.$$

Since each edge contains four 3-subsets, we have

$$\begin{aligned}|\mathcal{A}| &= \frac{1}{4}\left(\binom{n}{3} - \frac{n}{6}\binom{6}{3}\right)\\ &= \frac{1}{4}\binom{n}{3} - \frac{5n}{6}.\end{aligned}$$

If we let $\mathcal{B}_A = \binom{A}{3}$ for $A \in \mathcal{A}$ and for each group $G \in \mathcal{G}$, we let $\mathcal{B}_G(i) \subseteq \binom{G}{3}$ that is isomorphic to $\mathcal{A}_i$, $i \in [5]$, then the supplements of $(X, \mathcal{B}_A)$, $A \in \mathcal{A}$, and $(X, \mathcal{B}_G(i))$, $G \in \mathcal{G}$, $i \in [5]$ form an acyclic decomposition of the hypergraph $\left(X, \binom{X}{n-3}\right)$. The size of this acyclic decomposition is

$$|\mathcal{A}| + 5|\mathcal{G}| = \frac{1}{4}\binom{n}{3}.$$

Consequently, we have.

Proposition 5.2. *When $n \equiv 0 \pmod 6$, $\operatorname{arb}(K_n^{(n-3)}) = \frac{1}{4}\binom{n}{3}$.*

5.2. *The case $n \equiv 1 \pmod 2$*

Given a finite set X and a partition $\mathcal{V} = \{V_1, \ldots, V_r\}$ of X (that is, $\cup_{i\in[r]} V_i = X$ and $V_i \cap V_j = \emptyset$ for distinct $i, j \in [r]$), a set $Y \subseteq X$ with the property that $|Y \cap V_i| \le 1$ for all $i \in [r]$ is called a *transversal of X with respect to $\mathcal{V}$*.

A *k-uniform r-partite hypergraph* is a k-uniform hypergraph $\mathcal{H} = (X, \mathcal{A})$ with a partition $\mathcal{G} = \{G_1, \ldots, G_r\}$ of X into r parts G_i, such that for any $A \in \mathcal{A}$, A is a transversal of X with respect to $\mathcal{G}$. If $\mathcal{A}$ is the set of all transversals of X with respect to $\mathcal{G}$, then $(X, \mathcal{A})$ is called *complete*. The complete k-uniform r-partite hypergraph, where each part has size m, is denoted by $K_{r(m)}^{(k)}$.

Lemma 5.1. *Let $\mathcal{H}$ be a $K_{3(2)}^{(3)}$. Then $\operatorname{arb}(\mathcal{H}^s) = 2$.*

Proof. Consider the complete 3-uniform tripartite hypergraph $\mathcal{H} = ([6], \mathcal{A})$, with tripartition $\mathcal{G} = \{\{i, i+3\} : i \in [3]\}$, and let

$$\begin{aligned}\mathcal{A}_1 &= \{\{1, 2, 3\}, \{3, 4, 5\}, \{2, 3, 4\}, \{4, 5, 6\}\},\\ \mathcal{A}_2 &= \{\{1, 3, 5\}, \{1, 5, 6\}, \{2, 4, 6\}, \{1, 2, 6\}\}.\end{aligned}$$

The supplements of $([6], \mathcal{A}_1)$ and $([6], \mathcal{A}_2)$ are each acyclic, and they decompose $\mathcal{H}^s$.

Lemma 5.2. *For $n \in \{7, 9\}$, $\operatorname{arb}(K_n^{(n-3)}) = \lceil \frac{1}{4}\binom{n}{3} \rceil$.*

Proof. For $n = 7$, let $\mathcal{H} = \left(X, \binom{X}{3}\right)$, where $X = [7]$, and let

$$\begin{aligned}\mathcal{A}_1 &= \{\{1, 2, 7\}, \{1, 3, 7\}, \{2, 3, 7\}\},\\ \mathcal{A}_2 &= \{\{1, 4, 7\}, \{1, 5, 7\}, \{2, 4, 7\}, \{1, 2, 5\}\},\\ \mathcal{A}_3 &= \{\{1, 6, 7\}, \{2, 5, 7\}, \{2, 6, 7\}, \{1, 3, 6\}\},\\ \mathcal{A}_4 &= \{\{3, 4, 7\}, \{3, 5, 7\}, \{4, 5, 7\}, \{3, 4, 5\}\},\\ \mathcal{A}_5 &= \{\{3, 6, 7\}, \{4, 6, 7\}, \{1, 2, 3\}, \{2, 3, 6\}\},\\ \mathcal{A}_6 &= \{\{5, 6, 7\}, \{1, 2, 4\}, \{1, 2, 6\}, \{1, 5, 6\}\},\\ \mathcal{A}_7 &= \{\{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 4, 6\}\},\\ \mathcal{A}_8 &= \{\{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{4, 5, 6\}\},\\ \mathcal{A}_9 &= \{\{2, 4, 5\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 5, 6\}\}.\end{aligned}$$

The supplements of $([7], \mathcal{A}_i)$, $i \in [9]$, are each acyclic, and they decompose $\mathcal{H}^s = K_7^{(4)}$.

For $n = 9$, let $\mathscr{H} = \left(X, \binom{X}{3}\right)$, where $X = [9]$. The supplements of $([9], \pi^j(\mathscr{B}_i))$, for $i \in [3]$ and $j \in [7]$, where

$$\mathscr{B}_1 = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 5\}, \{2, 4, 5\}\},$$
$$\mathscr{B}_2 = \{\{1, 2, 5\}, \{1, 2, 8\}, \{1, 3, 8\}, \{2, 5, 9\}\},$$
$$\mathscr{B}_3 = \{\{1, 2, 9\}, \{1, 3, 9\}, \{1, 4, 8\}, \{4, 8, 9\}\},$$

and π is the permutation $(1\,2\,\ldots\,7)(8)(9)$, are each acyclic and they decompose $\mathscr{H}^s = K_9^{(6)}$. □

Lemma 5.3. *Let $\mathscr{H} = (X, \mathscr{A})$, where $(X, S, \mathscr{G}, \mathscr{A})$ is a (3,3)-CS$(2^2 : 1)$. Then* $\mathrm{arb}(\mathscr{H}^s) = 2$.

Proof. Take $X = [5]$, $S = \{5\}$, and $\mathscr{G} = \{\{i, i+2\} : i \in [2]\}$. Then $\mathscr{A}$ can be partitioned into

$$\mathscr{A}_1 = \{\{1, 2, 3\}, \{1, 2, 5\}, \{1, 4, 5\}, \{2, 3, 4\}\}, \quad \text{and}$$
$$\mathscr{A}_2 = \{\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 5\}, \{3, 4, 5\}\}.$$

Since the supplements of $(X, \mathscr{A}_1)$ and $(X, \mathscr{A}_2)$ are each acyclic, the lemma follows. □

Proposition 5.3. *When $n \equiv 1 \pmod 2$, $\mathrm{arb}(K_n^{(n-3)}) = \lceil \frac{1}{4}\binom{n}{3} \rceil$.*

Proof. First note that the proposition holds for $n \in \{1, 3, 5, 7, 9\}$ (it is trivial for $n \in \{1, 3\}$; $n = 5$ follows from our knowledge of $\mathrm{arb}(K_n^{(2)})$; $n \in \{7, 9\}$ follows from Lemma 5.2).

For $n \geq 11$, write $n = 8r + u$, where $1 \leq u \leq 7$. Let $X = ([r] \times [4] \times [2]) \cup S$, where

$$S = \begin{cases} \{\infty\}, & \text{if } u = 1 \\ \{\infty\} \cup ([(u-1)/2] \times [2]), & \text{otherwise.} \end{cases}$$

Let $Y = ([r] \times [4]) \cup S'$, where

$$S' = \begin{cases} \emptyset, & \text{if } u = 1 \\ [(u-1)/2], & \text{otherwise,} \end{cases}$$

and let $(Y, \mathscr{A})$ and $(Y, \mathscr{B})$ be hypergraphs $K^{(2)}_{4r+(u-1)/2}$ and $K^{(3)}_{4r+(u-1)/2}$, respectively.

The following acyclic decompositions are required:

(i) For each $i \in [r]$, let $X_i = (\{i\} \times [4] \times [2]) \cup \{\infty\}$. An acyclic decomposition $\{(X_i, \mathscr{A}_i(1)), (X_i, \mathscr{A}_i(2)), \ldots, (X_i, \mathscr{A}_i(21))\}$ of $K_9^{(6)}$ on vertex set X_i exists by Lemma 5.2.

(ii) An acyclic decomposition $\left\{(S, \mathscr{A}_0(1)), (S, \mathscr{A}_0(2)), \ldots, \left(S, \mathscr{A}_0\left(\lceil \frac{1}{4}\binom{u}{3} \rceil\right)\right)\right\}$ of $K_u^{(u-3)}$ on vertex set S exists.

(iii) For each $T \in \binom{Y}{3}$ not contained in S' or $\{i\} \times [4]$, $i \in [4]$, an acyclic decomposition $\{(T \times [2], \mathscr{B}_T(1)), (T \times [2], \mathscr{B}_T(2))\}$ of $K^{(3)}_{3(2)}$ on vertex set $T \times [2]$ with tripartition $\{\{x\} \times [2] : x \in T\}$ exists by Lemma 5.1.

(iv) For each $P \in \binom{Y}{2}$ not contained in S' or $\{i\} \times [4]$, $i \in [4]$, an acyclic decomposition $\{(P \times [2], \mathscr{C}_P(1)), (P \times [2], \mathscr{C}_P(2))\}$ of $(P \times [2], \mathscr{C})$, where $(P \times [2], \{\infty\}, \{\{x\} \times [2] : x \in P\}, \mathscr{C})$ is a (k, k)-CS$(2^2 : 1)$ exists by Lemma 5.3.

Let H be the set of hypergraphs

(i) $(X, \mathscr{A}_i(j))$, for $i \in [r]$ and $j \in [21]$,
(ii) $(X, \mathscr{A}_0(i))$, for $i \in [\lceil \frac{1}{4}\binom{u}{3} \rceil]$,
(iii) $(X, \mathscr{B}_T(i))$, for $T \in \binom{Y}{3} \setminus \left(\binom{S}{3} \cup \left(\cup_{i \in [r]} \binom{\{i\} \times [4]}{3}\right)\right)$ and $i \in [2]$,
(iv) $(X, \mathscr{C}_P(i))$, for $P \in \binom{Y}{2} \setminus \left(\binom{S}{2} \cup \left(\cup_{i \in [r]} \binom{\{i\} \times [4]}{2}\right)\right)$ and $i \in [2]$.

It is easy to check that the set $\{\mathscr{H}^s : \mathscr{H} \in H\}$ is an acyclic decomposition of $\left(X, \binom{X}{n-3}\right)$. The size of this acyclic decomposition is

$$21r + \left\lceil \frac{1}{4}\binom{u}{3} \right\rceil + 2\left(\binom{4r + \frac{u-1}{2}}{3} + \binom{4r + \frac{u-1}{2}}{2} - 4r - \binom{\frac{u-1}{2}}{3} - 6r - \binom{\frac{u-1}{2}}{2}\right) = \left\lceil \frac{1}{4}\binom{n}{3} \right\rceil. \quad \square$$

6. Conclusion

In this paper, techniques from combinatorial design theory are used to study the arboricity of complete uniform hypergraphs. As a result, the arboricity of the complete k-uniform hypergraph of order n is determined asymptotically when $k = n - O(\log^{1-\delta} n)$, δ positive, and determined exactly when $k = n - 3$. This proves a conjecture of Wang [20] in the asymptotic sense.

Note: Bermond et al. [4] have recently determined that $\mathrm{arb}(K_n^{(3)}) = \lceil n(n-1)/6 \rceil$, for all $n \geq 3$.

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