New Infinite Families of 2-Edge-Balanced Graphs

Cafer Caliskan[1] **and Yeow Meng Chee**[2]

[1]*Faculty of Engineering and Natural Sciences, Kadir Has University, Istanbul 34083, Turkey, E-mail: cafer.caliskan@khas.edu.tr*

[2]*Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore, E-mail: ymchee@ntu.edu.sg*

Received February 13, 2013; revised July 6, 2013

Published online 1 August 2013 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jcd.21367

Abstract: **A graph G of order n is called *t-edge-balanced* if G satisfies the property that there exists a positive λ for which every graph of order n and size t is contained in exactly λ distinct subgraphs of K_n isomorphic to G. We call λ the *index* of G. In this article, we obtain new infinite families of 2-edge-balanced graphs.** © 2013 Wiley Periodicals, Inc. J. Combin. Designs 22: 291–305, 2014

Keywords: *graphical t-designs; t-edge-balanced graphs*

1. INTRODUCTION

Our terminology and notation are standard (see [3] for undefined terms). We consider the problem of seeking a graph G of order n satisfying the property that there exists a positive λ for which every graph of order n and size t is contained in exactly λ distinct subgraphs of K_n isomorphic to G. We call such a graph G *t-edge-balanced*, and call λ its *index*. This problem is a special case of the problem of constructing graphical t-designs (all terms and notations are defined in the next section). Not every graph of order n is t-edge-balanced. For example, the graph of order n containing a star of order k and $n-k$ isolated vertices is not 2-edge-balanced for any $k \geq 2$, since it contains no pair of independent edges, and the graph of order $n \equiv 0 \pmod 2$ containing $n/2$ independent edges is not 2-edge-balanced since it contains no pair of incident edges. In fact, there has been only one explicit infinite family of 2-edge-balanced graphs known. Alltop [1]

Contract grant sponsor: National Research Foundation of Singapore; contract grant number: NRF-CRP2-2007-03; contract grant sponsor: Nanyang Technological University; contract grant number: M58110040.

has shown that when $n \geq 3$ is odd, the graph (of order n) containing a cycle of length $(n+3)/2$ and $(n-3)/2$ isolated vertices is 2-edge-balanced with index $\lambda = (n-3)!/((n-3)/2)!$.

For history and state-of-the-art results on t-edge-balanced graphs and graphical t-designs, we refer the reader to [4, 5].

The purpose of this paper is to provide an exposition of the method developed by Alltop [1] for finding 2-edge-balanced graphs and obtain new infinite families of 2-edge-balanced graphs. These also give rise to new infinite families of graphical 2-designs.

2. PRELIMINARIES

For a finite set X and a nonnegative integer t, the set of all t-subsets of X is denoted $\binom{X}{t}$. A *set system* is a pair $(X, \mathcal{A})$, where X is a finite set of elements called *points*, and $\mathcal{A} \subseteq 2^X$. Elements of $\mathcal{A}$ are called *blocks*. The *order* of $(X, \mathcal{A})$ is the number of points, $|X|$. A set system $(X, \mathcal{A})$ such that $\mathcal{A} \subseteq \binom{X}{k}$ is said to be *k-uniform*. A *t-design*, or more specifically a t-(v, k, λ) design, is a k-uniform set system $(X, \mathcal{A})$ of order v such that every $T \in \binom{X}{t}$ is contained in precisely λ blocks of $\mathcal{A}$. To avoid triviality, we impose the following restrictions on a t-(v, k, λ) design $(X, \mathcal{A})$:

(i) $t \geq 2$,
(ii) $t < k < v$,
(iii) $\mathcal{A} \neq \varnothing$, and $\mathcal{A} \neq \binom{X}{k}$.

For two set systems, $\mathcal{S}_1 = (X_1, \mathcal{A}_1)$ and $\mathcal{S}_2 = (X_2, \mathcal{A}_2)$, an *isomorphism* of $\mathcal{S}_1$ onto $\mathcal{S}_2$ is a bijection $\sigma : X_1 \to X_2$ such that $\sigma(\mathcal{A}_1) = \mathcal{A}_2$. A set system $\mathcal{S}_1$ is *isomorphic* to a set system $\mathcal{S}_2$, and written $\mathcal{S}_1 \cong \mathcal{S}_2$, if there exists an isomorphism of $\mathcal{S}_1$ onto $\mathcal{S}_2$. An *automorphism* of a set system is an isomorphism of the set system onto itself. The set of all automorphisms of a set system $\mathcal{S}$ forms a group under functional composition. This group is called *the automorphism group* of $\mathcal{S}$ and is denoted by Aut($\mathcal{S}$).

Let $V = V(K_n)$ be the set of vertices of the complete graph K_n on n vertices. The action of the symmetric group S_n on V also induces an action on $E = E(K_n) = \binom{V}{2}$, the set of edges of K_n. A t-$(\binom{n}{2}, k, \lambda)$ design $(E, \mathcal{A})$ is said to be *graphical* if it is fixed under the action of S_n, that is, $S_n(\mathcal{A}) = \mathcal{A}$. In particular, $\mathcal{A}$ is then a union of orbits of S_n on $\binom{E}{k}$. We can consider a subset $E' \subseteq E$ as a labeled graph with edge set E' and vertex set V. The orbits of S_n on 2^E are just the isomorphism classes of graphs on vertex set V, and therefore each such orbit can be represented by an unlabeled subgraph of K_n.

The connection between graphical t-designs and t-edge-balanced graphs is as follows: a graphical t-$(\binom{n}{2}, k, \lambda)$ design $(X, \mathcal{A})$ such that $\mathcal{A}$ contains a single orbit represented by G is equivalent to G being a graph of order n and size k that is t-edge-balanced with index λ. This equivalence is clear from the definitions of graphical t-designs and t-edge-balanced graphs.

Chee and Kaski [4] remarked that only a finite number of graphical t-designs are known. It came to our attention recently that an infinite family of 2-edge-balanced graphs, and hence graphical 2-designs, had already been discovered by Alltop [1] in 1966 (actually, this fact is also mentioned by Betten et al. [2] referenced in [4], but we had missed it).

3. ALLTOP'S METHOD

The essence of Alltop's method is the following elementary result, for which a proof is included for completeness.

Lemma 3.1 (Alltop [1]). *Let G and H be graphs of order n. Suppose G contains $n_{H:G}$ distinct subgraphs isomorphic to H. Then the number of distinct subgraphs of K_n isomorphic to G, each of which contains H, is*

$$\lambda_{H:G} = n_{H:G} \frac{|\mathrm{Aut}(H)|}{|\mathrm{Aut}(G)|}.$$

Proof. We count in two ways, N, the number of ordered pairs (H', G') satisfying the conditions

- H' is a subgraph of K_n isomorphic to H,
- G' is a subgraph of K_n isomorphic to G, and
- G' contains H'.

For a fixed H', there are $\lambda_{H:G}$ subgraphs of K_n isomorphic to G, each of which contains H'. Since the number of subgraphs of K_n isomorphic to H is $n!/|\mathrm{Aut}(H)|$, the total number of such ordered pairs (H', G') is

$$\lambda_{H:G} \frac{n!}{|\mathrm{Aut}(H)|}. \tag{1}$$

On the other hand, for a fixed G', G' contains $n_{H:G}$ subgraphs isomorphic to H. Since the number of subgraphs of K_n isomorphic to G is $n!/|\mathrm{Aut}(G)|$, the total number of such ordered pairs (H', G') is

$$n_{H:G} \frac{n!}{|\mathrm{Aut}(G)|}. \tag{2}$$

Equating (1) and (2) gives the required

$$\lambda_{H:G} = n_{H:G} \frac{|\mathrm{Aut}(H)|}{|\mathrm{Aut}(G)|}. \qquad \square$$

There are two isomorphism classes of graphs of order n and size two. These are shown in Fig. 1 (we adopt the convention that isolated vertices are not shown in graph drawings;

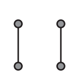
G	$H_1^{(2)}$	 $H_2^{(2)}$
$\mathrm{Aut}(G)$	$S_2 \times S_{n-3}$	$(S_2 \wr S_2) \times S_{n-4}$
$\lvert\mathrm{Aut}(G)\rvert$	$2(n-3)!$	$8(n-4)!$

FIGURE 1. Isomorphism classes of graphs of size two.

the order of the graphs involved should be clear from the context), together with their automorphism groups. A necessary and sufficient condition for a graph G to be 2-edge-balanced is $\lambda_{H_1^{(2)}:G} = \lambda_{H_2^{(2)}:G}$. It follows from Lemma 3.1 that this condition is equivalent to

$$n_{H_1^{(2)}:G}\left|\mathrm{Aut}\left(H_1^{(2)}\right)\right| = n_{H_2^{(2)}:G}\left|\mathrm{Aut}\left(H_2^{(2)}\right)\right|,$$

or

$$\frac{n_{H_2^{(2)}:G}}{n_{H_1^{(2)}:G}} = \frac{n-3}{4}.$$

We record this result as:

Theorem 3.1 (Alltop [1]). *A graph G of order n is 2-edge-balanced if and only if*

$$\frac{n_{H_2^{(2)}:G}}{n_{H_1^{(2)}:G}} = \frac{n-3}{4}.$$

Corollary 3.1 (Alltop [1]). *Let $k \geq 3$ and let G be the graph of order $2k-3$ and size k containing a cycle of length k. Then G is 2-edge-balanced of index $(2k-6)!/(k-3)!$.*

Proof. We have $n_{H_1^{(2)}:G} = k$ and $n_{H_2^{(2)}:G} = k(k-3)/2$. This gives $n_{H_2^{(2)}:G}/n_{H_1^{(2)}:G} = (k-3)/2 = (2k-6)/4$. It follows that G is 2-edge-balanced. The index of G follows from $|\mathrm{Aut}(G)| = 2k(k-3)!$. □

Corollary 3.2 (Alltop [1]). *There exists a graphical 2-$(\binom{2k-3}{2}, k, (2k-6)!/(k-3)!)$ design, for all $k \geq 3$.*

4. NEW INFINITE FAMILIES OF 2-EDGE-BALANCED GRAPHS

Let $S_{m,k}$ be a tree of size k and consisting of a vertex $v_0^{(S_{m,k})}$ of degree $m \geq 1$ and other vertices of degree 1 or 2. It is immediate that $k \geq m$. We label by $v_1^{(S_{m,k})}, v_2^{(S_{m,k})}, \ldots, v_m^{(S_{m,k})}$ the leaves of $S_{m,k}$. If we denote by $d_j^{(S_{m,k})}$ the distance of $v_j^{(S_{m,k})}$ from the vertex $v_0^{(S_{m,k})}$, note that $\sum d_j^{(S_{m,k})} = k$, where $d_j^{(S_{m,k})} \geq 1$. Based on the structure of the tree, we compute that $n_{H_1^{(2)}:S_{m,m}} = \binom{m}{2}$, and for a given m, $n_{H_1^{(2)}:S_{m,k}} = n_{H_1^{(2)}:S_{m,k-1}} + 1$ whenever $k > m$. Therefore,

$$n_{H_1^{(2)}:S_{m,k}} = k + \binom{m}{2} - m.$$

Moreover, $n_{H_2^{(2)}:S_{m,m}} = 0$, and for a given m, $n_{H_2^{(2)}:S_{m,k}} = n_{H_2^{(2)}:S_{m,k-1}} + k - 2$ whenever $k > m$, from which it follows that

$$n_{H_2^{(2)}:S_{m,k}} = \binom{k-1}{2} - \binom{m-1}{2}.$$

Let's define

$$N(m,k) = 4\frac{n_{H_2^{(2)}:S_{k,m}}}{n_{H_1^{(2)}:S_{k,m}}} + 3 = 4\frac{\binom{k-1}{2} - \binom{m-1}{2}}{k + \binom{m}{2} - m} + 3,$$

where $m, k \in \mathbb{Z}^+$. If $N = N(m,k)$ for some m and k, then define $G_{m,k}$ to be the union of $S_{m,k}$ and $N-k-1$ isolated vertices whenever $N \in \mathbb{Z}$ is at least $k+1$. Then it follows that $G_{m,k}$ is of order N. If $k = m$, except when $k = m = 2$, then $N < k+1$. Notice also that $G_{1,k} \cong G_{2,k}$. Thus, we assume that $k > m \geq 1$ throughout this article. Moreover, the size of the automorphism group of $G_{m,k}$ is computed as follows:

$$|Aut(G_{m,k})| = (N-k-1)!\ \Pi_{d=1}^{k-m+1}\left(\left|\left\{j \mid j \in \{1,\ldots,m\}, d_j^{(G_{m,k})} = d\right\}\right|\right)!$$

Theorem 4.1. *$G_{m,k}$ is 2-edge-balanced of index $\lambda = \frac{4k(k-1)(N-3)!}{(N+1)|Aut(G_{m,k})|}$ if and only if $N \in \mathbb{Z}$ is at least $k+1$.*

Proof. It is immediate by Theorem 3.1 and the index follows from the computation of the length of the orbit of $G_{m,k}$. □

Let's now consider the values of $m, k \in \mathbb{Z}^+$ such that $N \in \mathbb{Z}$ is at least $k+1$. Our computation shows that

$$N = \frac{4k^2 - 6k - A}{2k + A} = 2k - A - 3 + \frac{A(2+A)}{2k+A}, \tag{3}$$

where $A = m^2 - 3m$. Thus, we are interested in the values of $m, k \in \mathbb{Z}^+$ so that $N \geq k+1$ and $\frac{A(2+A)}{2k+A} \in \mathbb{Z}$, since $2k - A - 3 \in \mathbb{Z}$. Notice also that $2k + A \neq 0$, since $k > m \geq 1$. In particular, we let $A(2+A) = 0$, then the nonzero integer solutions are $m = 1, 2, 3$. This results in the following corollary.

Corollary 4.1. *$G_{m,k}$ is 2-edge-balanced for any $k > m$, where $m \in \{1,2,3\}$.*

In what follows, we first let $m \in \{1,2,3\}$ and then analyze the case $m \geq 4$.

4.1. $m = 1$ or $m = 2$

Let G be one of the graphs $G_{1,k}$ or $G_{2,k}$. Since $N(1,k) = N(2,k) = 2k-1$, there are exactly $k-2$ isolated vertices in G, so we have the following corollary.

Corollary 4.2. *For every $k > 1$, there is a graphical $2 - (\binom{2k-1}{2}, k, \frac{(k-1)(2k-4)!}{(k-2)!})$ design.*

4.2. $m = 3$

In this section, we consider the graph $G = G_{3,k}$. We partition the set of all such graphs into classes according to their automorphism groups. The automorphism group of G is S_{k-4} if $d_j^{(G)}$ are all distinct (Class I), and $S_3 \times S_{k-4}$ when $d_j^{(G)}$ are all equal (Class II). If exactly two of $d_j^{(G)}$ are equal (Class III), then the automorphism group of G is $S_2 \times S_{k-4}$.

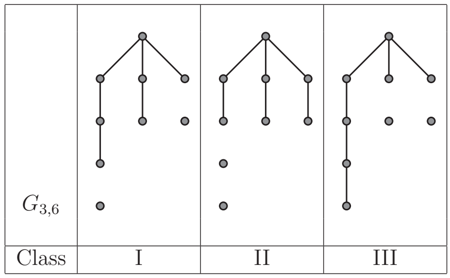


FIGURE 2. Graphs with $m = 3$ and $k = 6$.

See Fig. 2 for graphs of size 6 from different classes. If we let $G_{3,k}$ be in Class II, then we have the following corollary.

Corollary 4.3. *For any $k \geq 4$ divisible by 3, there is a graphical $2 - (\binom{2k-3}{2}, k, \frac{k(2k-6)!}{3(k-4)!})$ design.*

Let's now consider the graphs in Classes I and III, then we have the following result.

Corollary 4.4. *For any $k \geq 4$, there are graphical $2 - (\binom{2k-3}{2}, k, \frac{2k(2k-6)!}{(k-4)!})$ and $2 - (\binom{2k-3}{2}, k, \frac{k(2k-6)!}{(k-4)!})$ designs.*

In what follows, we compute the number of nonisomorphic graphs in Classes I and III. There is exactly one graph in Class II if $3 \mid k$, where $k \geq 4$ is the size of the graph.

Consider the equation

$$\sum d_j^{(G)} = k, \tag{4}$$

where $k \geq 4$ *and* $d_j^{(G)} \geq 1$. The total number of solutions for (4) is $\binom{k-1}{2}$. Let's now fix k and consider the solutions for (4), where there are exactly two of $d_j^{(G)}$ are equal, in the following cases:

Case 1. k is odd: Without loss of generality, assume that $d_1^{(G)} = d_2^{(G)}$ and $d_1^{(G)} \neq d_3^{(G)} \neq d_2^{(G)}$. Then,

$$\left|\left\{d_3^{(G)} : \frac{k - d_3^{(G)}}{2} \in \mathbb{Z},\ 1 \leq d_3^{(G)} \leq k - 2\right\}\right| = |\{1, 3, \ldots, k - 2\}| = \left\lceil \frac{k-2}{2} \right\rceil.$$

This implies that the number of nonisomorphic graphs in Class III is $\lceil \frac{k-2}{2} \rceil$ if $3 \nmid k$ and $\lceil \frac{k-2}{2} \rceil - 1$ if $3 \mid k$.

Case 2. k is even: Similarly, we determine that the number of non-isomorphic graphs in Class III is $\frac{k-2}{2}$ if $3 \nmid k$ and $\frac{k-2}{2} - 1$ if $3 \mid k$.

However, the total number of solutions for (4), under the condition that there are exactly two of $d_j^{(G)}$ are equal, is three times the number of non-isomorphic graphs in Class III. Note also that there are exactly six corresponding graphs in Class I for a single solution

TABLE I. Number of nonisomorphic graphs $G_{3,k}$ in different classes.

		Class I	Class II	Class III
$k \geq 4$ odd	$3 \mid k$	$\frac{\binom{k-1}{2} - 3\lceil \frac{k-2}{2} \rceil + 2}{6}$	1	$\lceil \frac{k-2}{2} \rceil - 1$
	$3 \nmid k$	$\frac{\binom{k-1}{2} - 3\lceil \frac{k-2}{2} \rceil}{6}$	0	$\lceil \frac{k-2}{2} \rceil$
$k \geq 4$ even	$3 \mid k$	$\frac{\binom{k-1}{2} - 3\frac{k-2}{2} + 2}{6}$	1	$\frac{k-2}{2} - 1$
	$3 \nmid k$	$\frac{\binom{k-1}{2} - 3\frac{k-2}{2}}{6}$	0	$\frac{k-2}{2}$

TABLE II. Some graphical 2-designs with $m = 3$ and small $k \geq 4$.

k	n	v	b	r	index λ	Class	# of graphical 2-designs
4	5	10	60	24	8	III	≥ 1
5	7	21	2,520	600	120	III	≥ 2
6	9	36	181,440	30,240	4,320	I	≥ 1
			30,240	5,040	720	II	≥ 1
			90,720	15,120	2,160	III	≥ 1

for (4). Thus, Table I provides with the number of nonisomorphic graphs in different classes and Table II the parameters for some graphical 2-designs with some small k.

4.3. $m \geq 4$

In this section, we focus on the following two questions:

1. Does there exist an integer-valued polynomial (function) K such that $G_{m,K}$ is 2-edge-balanced for any $m \geq 4$?
2. Does there exist a pair of integer-valued polynomials (functions) K and M such that $G_{M,K}$ is 2-edge-balanced whenever $M \geq 4$?

In this sense, we let $k = K$ in (3):

$$N = 2K - A - 3 + \frac{A(2 + A)}{2K + A},$$

where $A = m^2 - 3m$. We note that degree of $A(2 + A)$ is 4, then if the degree of K as a polynomial over m is at least 5, we let $m = 1, 2, 3$ and therefore $N = 2K - A - 3 \in \mathbb{Z}$ as we discuss above. In the following, we consider some polynomials K of degree at most 4 with the motivation of finding new families of 2-edge-balanced graphs.

4.3.1. Degree 1

Let $K = am + b \in \mathbb{Q}[m],\ a \neq 0$, then consider

$$N = \frac{4(a^2-1)m^2 + 4(2ab - 3a + 3)m + 4(b^2 - 3b)}{m^2 + (2a-3)m + 2b} + 3.$$

Thus, we have that

$$N \in \mathbb{Z} \text{ if and only if } 4(a^2-1) = \frac{4(2ab-3a+3)}{2a-3} = 2(b-3) \in \mathbb{Z},$$

for any m. This implies that $a \in \{-1, -\frac{1}{2}\}$. If $a = -1$, then $K = -m + 3 < 0$ for $m \geq 4$. Moreover, $N = 0$ if $a = -\frac{1}{2}$. Hence, there is not a pair of $a \neq 0, b \in \mathbb{Q}$ such that $G_{m,K}$ is 2-edge-balanced for any $m \geq 4$. However, there may still be some values for a, b that result in 2-edge-balanced graphs for certain m values. Among many examples, we provide with some examples of polynomials K that satisfy our conditions. Realize that K is of degree 1 over m and degree 2 over the parameter t.

(i) Let $a = 1 + t$, $b = -1 - 2t$ and $M_1 = 1 + 2t$, $t \in \mathbb{Z}^+ \setminus \{1\}$, then we have that

$$\begin{aligned} K_1 &= 2t^2 + t, \\ N_1 &= 2t^2 + 2t - 1 \in \mathbb{Z}, \quad \text{and} \\ N_1 - K_1 &= t - 1 \geq 1. \end{aligned}$$

(ii) Let $a = 1 + t$, $b = -t$ and $M_2 = 2t$, $t \in \mathbb{Z}^+ \setminus \{1\}$, then we have that

$$\begin{aligned} K_2 &= 2t^2 + t, \\ N_2 &= 2t^2 + 3t \in \mathbb{Z}, \quad \text{and} \\ N_2 - K_2 &= 2t \geq 1. \end{aligned}$$

4.3.2. Degree 2

Let $K = am^2 + bm + c \in \mathbb{Q}[m],\ a \neq 0$, then

$$N = Q + \frac{R_1 m + R_0}{D} + 3,$$

where

$$R_0 = \frac{2(2a+1)c^2 - (22a^2 + 12ab + 2b^2 + 4a + 1)c}{8a^3 + 12a^2 + 6a + 1}, \quad \text{and}$$

$$R_1 = \frac{-2(3(4a-1)b^2 + 2a^3 - 15a^2 + 2b^3 + (22a^2 - 14a + 1)b - 2((2a+1)b + 6a^2 + 3a)c + 3a}{8a^3 + 12a^2 + 6a + 1}.$$

Set $R_0 = 0$, then one solution is that $c = 0$. Thus, we substitute $c = 0$ in $R_1 = 0$. This gives rise to two solutions for b, namely $b = 1 - a$ and $b = -3a$. Another solution for $R_0 = 0$ is that $c = (22a^2 + 12ab + 2b^2 + 4a + 1)/(2a + 1)$. This solution implies that $b = -3a$ or $b = -5a - 1$ in $R_1 = 0$. In the following we discuss some polynomials with coefficients based on these solutions.

(i) $K = am^2 + (1 - a)m, a \in \mathbb{Z}^+$.
Let $m \equiv 2$ or 3 $(mod\ 2a + 1)$, where $a \in \mathbb{Z}^+$, then we write $M_3 = 2 + (2a + 1)t$ and $M_4 = 3 + (2a + 1)t$, where $t \in \mathbb{Z}^+$. Then we compute that

$$\begin{aligned} K_3 &= (4a^3 + 4a^2 + a)t^2 + (6a^2 + 5a + 1)t + 2a + 2, \\ N_3 &= 4(2a^3 + a^2)t^2 + 4(3a^2 + 2a)t + 4a + 3 \ \in \mathbb{Z}, \quad \text{and} \\ N_3 - K_3 &= (4a^3 - a)t^2 + (6a^2 + 3a - 1)t + 2a + 1 \geq 1. \end{aligned}$$

and

$$\begin{aligned} K_4 &= (4a^3 + 4a^2 + a)t^2 + (10a^2 + 7a + 1)t + 6a + 3, \\ N_4 &= 4(2a^3 + a^2)t^2 + 4(5a^2 + 2a)t + 12a + 3 \ \in \mathbb{Z}, \quad \text{and} \\ N_4 - K_4 &= (4a^3 - a)t^2 + (10a^2 + a - 1)t + 6a \geq 1. \end{aligned}$$

(ii) $K = am^2 - 3am, a \in \mathbb{Z}^+$.
Let $m \equiv 1$ or 2 $(mod\ 2a + 1)$, where $a \in \mathbb{Z}^+$, then we write $M_5 = 1 + (2a + 1)t$, where $t \in \mathbb{Z}^+$ except when $a = t = 1$, and $M_6 = 2 + (2a + 1)t$, where $t \in \mathbb{Z}^+$. Then we have that

$$\begin{aligned} K_5 &= (4a^3 + 4a^2 + a)t^2 - (2a^2 + a)t - 2a, \\ N_5 &= 4(2a^3 + a^2)t^2 - 4a^2t - 4a - 1 \ \in \mathbb{Z}, \quad \text{and} \\ N_5 - K_5 &= (4a^3 - a)t^2 - (2a^2 - a)t - 2a - 1 \geq 1. \end{aligned}$$

and

$$\begin{aligned} K_6 &= (4a^3 + 4a^2 + a)t^2 + (2a^2 + a)t - 2a, \\ N_6 &= 4(2a^3 + a^2)t^2 + 4a^2t - 4a - 1 \ \in \mathbb{Z}, \quad \text{and} \\ N_6 - K_6 &= (4a^3 - a)t^2 + (2a^2 - a)t - 2a - 1 \geq 1. \end{aligned}$$

(iii) $K = am^2 - 3am + 2a + 1, a \in \mathbb{Z}^+$.
Let $m \equiv 0$ or 3 $(mod\ 2a + 1)$, where $a \in \mathbb{Z}^+$, then we write $M_7 = (2a + 1)t$, where $t \in \mathbb{Z}^+$ except when $a = t = 1$, and $M_8 = 3 + (2a + 1)t$, where $t \in \mathbb{Z}^+$. Then we compute that

$$\begin{aligned} K_7 &= (4a^3 + 4a^2 + a)t^2 - 3(2a^2 + a)t + 2a + 1, \\ N_7 &= 4(2a^3 + a^2)t^2 - 12a^2t + 4a - 1 \ \in \mathbb{Z}, \quad \text{and} \\ N_7 - K_7 &= (4a^3 - a)t^2 - 3(2a^2 - a)t + 2a - 2 \geq 1. \end{aligned}$$

and

$$\begin{aligned} K_8 &= (4a^3 + 4a^2 + a)t^2 + 3(2a^2 + a)t + 2a + 1, \\ N_8 &= 4(2a^3 + a^2)t^2 + 12a^2t + 4a - 1 \ \in \mathbb{Z}, \quad \text{and} \\ N_8 - K_8 &= (4a^3 - a)t^2 + 3(2a^2 - a)t + 2a - 2 \geq 1. \end{aligned}$$

(iv) $K = am^2 - (5a+1)m + \frac{24a+3}{2a+1}$, $a \in \mathbb{Z} \setminus \{0\}$.
Let's assume one of the following cases for a and m:

(i) $(a, m) \in \{(-5, 4), (-5, 5), (-2, 4), (-2, 5), (-1, 4), (-1, 5), (-1, 6)\}$.
(ii) $(a, m) \in \{(1, m) \mid m \geq 6\}$.
(iii) $(a, m) \in \{(4, m) \mid m \geq 5\}$.

It follows that $K \in \mathbb{Z}$ is at least $m + 1$ for any $m \geq 4$. However, we are interested in the cases satisfying that $N \in \mathbb{Z}$ is at least $k + 1$. If $(a, m) = (-2, 5)$, then $N = 11$. Let us now assume that $a = 1$ and $m \equiv 0$ or 1 $(mod\ 3)$, where $m \geq 9$. If we write $M_9 = 3t$ or $M_{10} = 1 + 3t$, then $t \in \mathbb{Z}^+ \setminus \{1, 2\}$. Then we have that

$$\begin{aligned} K_9 &= 9t^2 - 18t + 9, \\ N_9 &= 12t^2 - 28t + 15 \ \in \mathbb{Z}, \quad \text{and} \\ N_9 - K_9 &= 3t^2 - 10t + 6 \geq 1. \end{aligned}$$

and

$$\begin{aligned} K_{10} &= 9t^2 - 12t + 4, \\ N_{10} &= 12t^2 - 20t + 7 \ \in \mathbb{Z}, \quad \text{and} \\ N_{10} - K_{10} &= 3t^2 - 8t + 3 \geq 1. \end{aligned}$$

Let's consider the case that $(a, m) = (-2, 5)$, then we have the following results.

Corollary 4.5. *$G_{5,10}$ is 2-edge-balanced.*

Corollary 4.6. *There exists a graphical* $2 - (55, 10, \frac{30 \cdot 8!}{|Aut(G_{5,10})|})$ *design.*

4.3.3. Degree 3
Let $K = am^3 + bm^2 + cm + d \in \mathbb{Q}[m]$, $a \neq 0$, then

$$N = N(m, K) = Q + \frac{R_2 m^2 + R_1 m + R_0}{D} + 3,$$

where

$$\begin{aligned} R_0 &= \frac{(12a + 2b + 1)d}{4a^2}, \\ R_1 &= \frac{2(12a + 2b + 1)c - 24a^2 - 4ad - 36a - 6b - 3}{8a^2} \quad \text{and} \\ R_2 &= \frac{4(6a + 1)b + 44a^2 - 4ac + 4b^2 + 18a + 1}{8a^2}. \end{aligned}$$

Set $R_0 = 0$, then one solution stems from the equation $12a + 2b + 1 = 0$. Thus, we substitute $b = (-1 - 12a)/2$ in $R_1 = 0$ and $R_2 = 0$. This implies that $c = 11a + 3/2$ and $d = -6a$. Let $M_{11} = 2at$, where $a, t \in \mathbb{Z}^+$ except when $(a, t) \in \{(1, 1), (1, 2)\}$, then we compute that

$$\begin{aligned} K_{11} &= 8a^4t^3 - 2(12a^3 + a^2)t^2 + (22a^2 + 3a)t - 6a, \\ N_{11} &= 16a^4t^3 - 8(6a^3 + a^2)t^2 + (44a^2 + 12a + 1)t - 12a - 3 \in \mathbb{Z}, \quad \text{and} \\ N_{11} - K_{11} &= 8a^4t^3 - 6(4a^3 + a^2)t^2 + (22a^2 + 9a + 1)t - 6a - 3 \geq 1. \end{aligned}$$

Another solution for $R_0 = 0$ is that $d = 0$. Thus, we let $d = 0$ in $R_1 = 0$ and $R_2 = 0$ and this results in three sets of solutions where $a \neq 0$:

(i) Let $b = -5a - 1/2$, $c = 6a + 3/2$, and $M_{12} = 1 + 2at$, where $a, t \in \mathbb{Z}^+$ except when $a = t = 1$, then

$$\begin{aligned} K_{12} &= 8a^4t^3 - 2(4a^3 + a^2)t^2 - (2a^2 - a)t + 2a + 1, \\ N_{12} &= 16a^4t^3 - 8(2a^3 + a^2)t^2 - (4a^2 - 4a - 1)t + 4a + 1 \in \mathbb{Z}, \quad \text{and} \\ N_{12} - K_{12} &= 8a^4t^3 - 2(4a^3 + 3a^2)t^2 - (2a^2 - 3a - 1)t + 2a \geq 1. \end{aligned}$$

(ii) Let $b = -4a - 1/2$, $c = 3a + 3/2$, and $M_{13} = 2 + 2at$, where $a, t \in \mathbb{Z}^+$, then

$$\begin{aligned} K_{13} &= 8a^4t^3 + 2(4a^3 - a^2)t^2 - (2a^2 + a)t - 2a + 1, \\ N_{13} &= 16a^4t^3 + 8(2a^3 - a^2)t^2 - (4a^2 + 4a - 1)t - 4a + 1 \in \mathbb{Z}, \quad \text{and} \\ N_{13} - K_{13} &= 8a^4t^3 + 2(4a^3 - 3a^2)t^2 - (2a^2 + 3a - 1)t - 2a \geq 1. \end{aligned}$$

(iii) Let $b = -3a - 1/2$, $c = 2a + 3/2$, and $M_{14} = 3 + 2at$, where $a, t \in \mathbb{Z}^+$, then

$$\begin{aligned} K_{14} &= 8a^4t^3 + 2(12a^3 - a^2)t^2 + (22a^2 - 3a)t + 6a, \\ N_{14} &= 16a^4t^3 + 8(6a^3 - a^2)t^2 + (44a^2 - 12a + 1)t + 12a - 3 \in \mathbb{Z}, \quad \text{and} \\ N_{14} - K_{14} &= 8a^4t^3 + 6(4a^3 - a^2)t^2 + (22a^2 - 9a + 1)t + 6a - 3 \geq 1. \end{aligned}$$

4.3.4. Degree 4

Let $K = am^4 + bm^3 + cm^2 + dm + e \in \mathbb{Q}[m]$, $a \neq 0$, then

$$N = N(m, K) = Q + \frac{R_3m^3 + R_2m^2 + R_1m + R_0}{D} + 3,$$

where

$$R_0 = -\frac{e}{2a}, \qquad R_1 = -\frac{12a + 2d - 3}{4a},$$

$$R_2 = \frac{22a - 2c - 1}{4a} \qquad \text{and} \qquad R_3 = -\frac{6a + b}{2a}.$$

We set $R_i = 0$ and this implies that $b = -6a$, $c = (-1 + 22a)/2$, $d = (3 - 12a)/2$, and $e = 0$. However, these assumptions do not give rise to a particular set of m values so that our requirements are satisfied. In the following we provide with some examples of monic polynomials K that result in some graphical 2-designs for certain m values. However, we note that in part (i) and (iii) K is of degree 2 over the parameter t although it is of degree 4 over m.

(i) Let $a = 1$, $b = -3 - 3t$, $c = -b$, $d = e = 0$, and $M_{15} = 2 + 3t$, $t \in \mathbb{Z}^+$, then we have that

$$\begin{aligned} K_{15} &= 9t^2 + 12t + 4, \\ N_{15} &= 12t^2 + 20t + 7 \in \mathbb{Z}, \quad \text{and} \\ N_{15} - K_{15} &= 3t^2 + 8t + 3 \geq 1. \end{aligned}$$

(ii) Let $a = 1$, $b = -3 - 3t$, $c = -b$, $d = e = 0$, and $M_{16} = 1 + 4t$, $t \in \mathbb{Z}^+$, then we compute that

$$\begin{aligned} K_{16} &= 64t^4 - 32t^3 - 12t^2 + 4t + 1, \\ N_{16} &= 128t^4 - 64t^3 - 40t^2 + 12t + 3 \in \mathbb{Z}, \quad \text{and} \\ N_{16} - K_{16} &= 64t^4 - 32t^3 - 28t^2 + 8t + 2 \geq 1. \end{aligned}$$

(iii) Let $a = 1$, $b = -4 - 3t$, $c = -b$, $d = e = 0$, and $M_{17} = 3 + 3t$, $t \in \mathbb{Z}^+$, then

$$\begin{aligned} K_{17} &= 9t^2 + 18t + 9, \\ N_{17} &= 12t^2 + 28t + 15 \in \mathbb{Z}, \quad \text{and} \\ N_{17} - K_{17} &= 3t^2 + 10t + 6 \geq 1. \end{aligned}$$

Let K_i and M_i, $i \in \{1, \ldots, 17\}$, be as above. If $i \in \{1, 2, 9, 10, 15, 16, 17\}$ and $t \in \mathbb{Z}^+$ except when $t = 1$ for $i \in \{1, 2, 9, 10\}$ and $t = 2$ for $i \in \{9, 10\}$, then we have that

Corollary 4.7. *$G_{M_i(t),K_i(t)}$ is 2-edge-balanced.*

Corollary 4.8. *There are infinite families of graphical 2-designs.*

If $i \in \{3, 4, 5, 6, 7, 8, 11, 12, 13, 14\}$ and $a, t \in \mathbb{Z}^+$ except when $a = t = 1$ for $i \in \{5, 8, 12\}$, then we have that

Corollary 4.9. *$G_{M_i(a,t),K_i(a,t)}$ is 2-edge-balanced.*

Corollary 4.10. *There are infinite families of polynomials each of which results in infinite families of graphical 2-designs.*

5. FURTHER RESULTS ON 2-EDGE-BALANCED GRAPHS

Let G be a graph of order n and size k. Then, $n_{H_1^{(2)}:G} + n_{H_2^{(2)}:G} = \binom{k}{2}$, from which it follows that

$$\frac{n_{H_2^{(2)}:G}}{n_{H_1^{(2)}:G}} = \frac{\binom{k}{2} - n_{H_1^{(2)}:G}}{n_{H_1^{(2)}:G}}.$$

TABLE III. Some m, k values resulting in 2-edge-balanced graphs $G_{m,k}$.

m	$k \leq 10000$
4	10
5	10, 15, 25, 55
6	21, 27, 36, 51, 81, 171
7	21, 28, 46, 56, 70, 91, 126, 196, 406
8	36, 40, 50, 64, 85, 100, 120, 148, 190, 260, 400, 820
9	36, 45, 57, 81, 99, 141, 162, 189, 225, 351, 477, 729, 1485
10	49, 55, 70, 85, 91, 105, 133, 145, 175, 217, 245, 280, 325, 385, 469, 595, 805, 1225, 2485
11	55, 66, 76, 88, 121, 136, 154, 176, 220, 286, 316, 352, 396, 451, 616, 748, 946, 1276, 1936, 3916
12	78, 81, 111, 126, 144, 166, 216, 243, 276, 342, 441, 486, 540, 606, 936, 1134, 1431, 1926, 2916, 5886
13	78, 91, 100, 130, 155, 195, 221, 265, 325, 364, 507, 595, 650, 715, 793, 1365, 1651, 2080, 2795, 4225, 8515
14	105, 154, 196, 209, 231, 287, 352, 385, 469, 495, 781, 847, 924, 1015, 1639, 1925, 2926, 3927, 5929
15	105, 120, 144, 162, 170, 183, 225, 274, 300, 330, 365, 378, 456, 495, 540, 690, 729, 820, 1002, 1080, 1170, 1275, 1548, 1730, 2250, 2640, 3186, 4005, 5370, 8100

For G to be 2-edge-balanced, we require

$$n = \frac{4(\binom{k}{2} - n_{H_1^{(2)}:G})}{n_{H_1^{(2)}:G}} + 3$$

to be an integer. Hence, $n_{H_1^{(2)}:G}$ must divide $2k(k-1)$. Based on this condition, we present some 2-edge-balanced graphs G in Table IV in which we adopt the following notation:

Graph-theoretic:

E_n —empty graph (graph of order n with no edges).

P_n —path of length n.

C_n —cycle of length n.

Group-theoretic:

D_n —dihedral group of order $2n$.

6. CONCLUSION

In this article, we show the existence of new infinite families of 2-edge-balanced graphs. Table III lists all m, k ($4 \leq m \leq 15$, $m < k \leq 10,000$) values such that $G_{m,k}$ is

TABLE IV. Some 2-edge-balanced graphs of order *n* and size *k*.

G	$n_{H_1^{(2)}:G}$	n, k	$\mathrm{Aut}(G)$	index λ
$P_2 \cup (k-2)P_1 \cup E_{2k(k-2)}$	1	$2k(k-1)-1, k \geq 2$	$(S_2)^{k-1} \times S_{k-2} \times S_{2k(k-2)}$	$\frac{2^{(2-k)}(2k^2-2k-4)!}{(k-2)!(2k^2-4k)!}$
$2P_2 \cup (k-4)P_1 \cup E_{k^2-3k+1}$	2	$k(k-1)-1, k \geq 3$	$(S_2)^{k-1} \times S_{k-4} \times S_{k^2-3k+1}$	$\frac{2^{(3-k)}(k^2-k-4)!}{(k-4)!(k^2-3k+1)!}$
$P_3 \cup (k-3)P_1 \cup E_{k^2-3k+1}$	2	$k(k-1)-1, k \geq 3$	$(S_2)^{k-2} \times S_{k-3} \times S_{k^2-3k+1}$	$\frac{2^{(4-k)}(k^2-k-4)!}{(k-3)!(k^2-3k+1)!}$
$C_4 \cup (k-4)P_1 \cup E_{(k-2)(k-3)/2}$	4	$k(k-1)/2-1, k \geq 4$	$D_4 \times (S_2)^{k-4} \times S_{k-4} \times S_{(k-2)(k-3)/2}$	$\frac{2^{(4-k)}(1/2k^2-1/2k-4)!}{(k-4)!(1/2k^2-5/2k+3)!}$
$P_5 \cup (k-4)P_1 \cup E_{(k^2-5k+2)/2}$	4	$k(k-1)/2-1, k \geq 5$	$(S_2)^{k-3} \times S_{k-4} \times S_{(k^2-5k+2)/2}$	$\frac{2^{(6-k)}(1/2k^2-1/2k-4)!}{(k-4)!(1/2k^2-5/2k+1)!}$
$P_k \cup E_{k-2}$	$k-1$	$2k-1, k \geq 2$	$S_2 \times S_{k-2}$	$\frac{(k-1)(2k-4)!}{(k-2)!}$

2-edge-balanced. A natural question is whether the Alltop's method [1] could be developed to obtain t-edge-balanced graphs ($t \geq 3$) and produce new infinite families of t-edge-balanced graphs.

REFERENCES

[1] W. O. Alltop, On the construction of block designs, J Combin Theory 1 (1966), 501–502.

[2] A. Betten, M. Klin, R. Laue, and A. Wassermann, Graphical t-designs via polynomial Kramer–Mesner matrices, Discrete Math 197/198 (1999), 83–109.

[3] B. Bollobas, Modern Graph Theory, vol. 184 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1998.

[4] Y. M. Chee and P. Kaski, An enumeration of graphical designs, J Combin Des 16 (2008), 70–85.

[5] Y. M. Chee and D. L. Kreher, "Graphical designs," in The CRC Handbook of Combinatorial Designs, C. J. Colbourn, J. H. Dinitz (Editors), 2nd ed., CRC Press, Boca Raton, 2007, pp. 490–493.